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In chapter 3 of our paper we present equations of motion for continuous mass distribution subject to hydrodynamic forces in their most general form. We start with equations for discrete mass particles and then transform the equations so that it is appropriate for a continuous mass distribution. As we do that, new forms of interactions are generated and we successfully include these interactions, using the propagator theory, in the general form of our hydrodynamic equations for continuous mass distributions. We also took a deeper mathematical description of rotational flows. We were able to explain many physical phenomena successfully by our treatment of rotational flows in a more concrete and simple way, for example, the phenomenon of ripples that appear on ocean beaches and in desert sands.

In chapter 4 we study the behavior of water surfaces. A liquid drop of water takes on a spherical shape because of the phenomenon of surface tension. A physical model based on the arrangement which the water molecules have on the surface is introduced to explain the above phenomenon. A mathematical model, as well as the physical model mentioned above, is introduced to describe the kind of forces involved on a wavy surface. The equations obtained describe the phenomenon of surface tension on a microscopic level very successfully.

In chapter 5 we apply the results of chapters 3 and 4 to get an equation that gives a critical dynamical value which govern the interactions between the moving fluid and the dust particles residing on the ground.

The Hydrodynamic Theory of Mass Transport and Matter Forces of Water

by

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I understand that my thesis will become part of the permanent collection of Oregon State University libraries. My signature below authorizes release of my thesis to any reader upon request.

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Abdulmuhsen H. Ali, Author

DEDICATION

To the Nation with good deeds.

② ACKNOWLEDGEMENTS

I would like to note that this book was a result of many years and hours of dedication and serious work. It is important that the reader take as much time as possible to go carefully through the details to develop a deep and a good understanding of the subject.

This scientific work is in an attempt to have a better approach and understanding of nature, as it is vital to have a good understanding of nature to interact with it properly.

Due to the help I received from the Mathematics Department and the Physics Department I was able to accomplish this theoretical work and recover the one year of research that was wasted due to Philip J. Siemens mismanagement and misleading advice that almost terminated my academic career, which makes me believe that the Physics Department needs to make a serious reevaluation of that faculty member. I consider this theoretical work to be a new step toward a better theory making in the field of fluid dynamics. I would like to thank Professor Ronald B. Guenther for his great help and understanding during the time we were putting the theory together. I was amazed by his deep understanding of the foundations of Physics in addition to his Mathematical expertise. Professor Ronald B. Guenther is of a great value to science and human history as his realization of a supporting philosophy in any scientific work proved his unique and effective method to approach nature. Also I would like to thank Kuwait University for their support and help during all the years I spent as a graduate student. Their realization of the significance of academic research is greatly appreciated. Many thanks to Paul Palmer for using his expertise patiently to type this book. Special thanks to Professors Henry Jansen, David Griffiths, Tom Giebultowicz, Lary Chen, and Robert Burton for serving on my committee and making valuable comments.

Finally, I would like to thank Kuwait University one more time for their valuable understanding and support during all the years I spent in the United States. It would have been impossible for me to do any kind of scientific work overseas without the financial help I received from Kuwait University.

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1 Introduction

Mathematical models of physical phenomena are simplified descriptions of nature. As we try to include more aspects of a given phenomenon into our model, the model itself becomes increasingly more complicated. In chapter 3 of our paper we develop mathematical equations that model the transport of point masses (dust particles) in fluids. We include rotational motion into our formalism and so incorporate turbulent flow into our hydrodynamic theory of mass transport. In chapter 4 we make use of statistical methods to introduce an equation that models the surface tension of water. In chapter 5 we apply these results obtained in chapters 3 and 4. We define a critical dynamical variable at which the interaction between the turbulent water and the residing dust occur and give a mathematical expression for that variable. We begin by giving a more detailed discussion of the results derived in chapter 3.

The problem of turbulent and wavy flow is a complex one and consequently the mathematical model describing mass transport in turbulent flows are themselves complex. Our system consists of water interacting with the ground, exciting the dust residing on it, setting the dust into motion, and finally the dust settling out over time to come to rest once more on the ground. We begin by considering curved streamlines (see D. S. Chandrasekharaiah and L. Debuath [6]) that curve downward toward the ground and scatter dust particles. Each streamline hits a specific dust particle, thereby setting it into motion. We assign radial, polar, and azimuthal coordinates to the excited particles to keep track of its motion and use

Lagrangian formalism to write the Newtonian force equation in terms of these coordinates. The rest of chapter 3 is dedicated to the question of the kind of forces applied to these dust particles and their coordinates.

We distinguish two types of forces the first type which is the delivered force at the instant of scattering, and the second type which is the active force after the excitation or the scattering takes place. The application of the first type of force during the scattering is assumed to be instantaneous. Using the concept of conservation of momentum, we relate the gain in momentum of the dust particles to the history of the scattering streamline. The type 2 force as we mentioned above is the sum of the applied (active) forces on the scattered dust particles due to the turbulent motion of the fluid and the presence of other previously scattered particles. In our search for an expression for this second type of force we found valuable mathematical expressions and dynamical quantities. One such dynamical quantity was a new vector potential (with symbol $\vec{A}$) that accounts for the rotational flow and obeys wave equation. As a result of that, $\vec{A}$ was an oscillatory function. The different physical phenomena we were able to explain in terms of this new $\vec{A}$ is an indication of its intrinsic existence. For example, the phenomenon of sand ripples on ocean beaches we were able to explain in terms of $\vec{A}$, the presence of which generates the ripple pattern. In fact our theory is also applicable to general viscous fluids flowing over a sandy medium. In his paper, P. Blondeaux [4] introduced a predictive theory for the formation of sand ripples under sea water. His theory was based on an analysis of a flat, sandy bottom subject to a viscous oscillatory flow. He did not, however, give a deep reason that accounts for the oscillatory flow, hence oscillatory pattern on the ground or ripples. In our paper we resolve this gap by use of the new vector potential $\vec{A}$, which explains the phenomenon of ripples in terms of an oscillatory drag force that causes the ripples. On the other end of the problem is the assumption of a preexisting wavy surface that is being subjected to wind flow. The wind doesn't have to be in an oscillatory flow as in Blondeaux, however, the existing stationary system of ripples create such a condition and the phenomenon of ripples can then be studied. This is what Theodore von Karman [29] does in his two attempts to resolve the phenomenon of ripples. However, we have the same shortcomings in his paper as we did in Blondeaux. He never gave a deep reason or cause for the initial formation of the ripples. The phenomenon of ripples occurs even if we start with a flat sandy surface. Besides, he assumes in his first method of approach that the density of air is negligible compared to the density of sand particles. This limits his solution to wind flows only and we cannot apply to water. Our theory describes the influence of any continuous media on its bounding surfaces, and avoids the assumption of preexisting ripples. We therefore generalize the phenomenon of ripple formation in the sense that we only need to have a flowing continuous media over a surface of particles to observe that periodic pattern. Karman mentioned in his paper that the ripples' arrangement are regular. This perfect regularity of ripples that Karman claims no satisfactory explanation has been offered for, is a result we obtained in our theory due to the fact that the vector potential $\vec{A}$, responsible for rotational flow, obeys a wave equation. We also mentioned the role of the vector potential in the presence or absence of the no slip condition on boundaries. In his article J. Serrin [25] explains Stokes' argument for the no slip condition. He also gives an equation, with an undetermined function, that corresponds to a slip condition. In contrast, we give an equation that gives an oscillatory tangential velocity at the boundary and recovers the no slip condition at a certain wavelength of the incoming water waves. This demonstrates our success in resolving the boundary condition problem that others, as Serrin mentioned, like Maxwell, Duhem, Knudsen, Chang and Uhlenbeck, Truesdell, and Patterson proposed solutions for.

Since our second type of force is expressed in terms of a canonical momentum density, which we can derive from a Lagrangian density, we must find equations of motion that govern the fluid velocity potential fields. We can then construct Lagrangian densities that correspond to the equations of motion of the fields. In order to have nonzero canonical momentum densities, we express our Lagrangian density in terms of the potentials as well as their time derivative. Any other interactions that are left out can then be incorporated into the second type force by adding it as an interaction Lagrangian density.

At this point of the theory mathematical modeling takes over as the starting point in the quest to understand natural phenomena. The typical ancient method is to start with a philosophical model that reflects our understanding and perception of a specific natural phenomenon and then to formulate that physical observation or philosophical perception of nature in terms of a mathematical equation. We trade the above privilege of the ancient approach to theory making with the privilege of flexibility in our theory with respect to modeling. Modeling can be done in our case by substituting a different interaction term than the one we proposed. The task of choosing a different or a "better" interaction term, however, is made easier by our invention of the new vector potential $\vec{A}$ which obeys a wave equation. The task is made easier because in addition to the homogeneous wave equation of $\vec{A}$, we also have a nonhomogeneous differential equation for $\vec{A}$ with a force dependent term on its right hand side. The best interaction Lagrangian density would then be the one which yields the most compatible force term on the right hand side of the nonhomogeneous differential equation of $\vec{A}$ when substituting the overall Lagrangian density into the Euler-Lagrange equation of motion.

This concludes our theoretical modeling in chapter 3; however, the above modeling was for discrete streamlines and particles. As we took the resulting equations to a continuous limit new complications emerged. We introduced the concept of a water beam, which is a continuous collection of streamlines. Then we rewrote our equations to describe a multi-beam picture. As different beams hit the ground, their corresponding streamlines bounce off the ground and interact with each other as well as with the particles which have been already scattered and diffused into the fluid. To include these new interactions we proposed a brilliant new particle flux that has a direct proportionality to the square of the concentration and used the continuity equation for particles based on the assumption that the motion of the particles is governed entirely by the fluid motion. This assumes fine dust particles, or, violent water motion such that the particles are driven completely by the motion of the fluid. Accordingly, we invented another new potential Φ, this time scalar, the gradient of which gives the particles' velocity in the diffusion equation. We showed that the continuity equation of Φ yields an equation that makes the application of the propagator theory possible. Through the propagator theory we were able to include all the interactions that are due to the beam interferences. This success in the theoretical modelling of the particle's flux is supported by G. J. Kynch [18] as he mentions in his paper that the particle's flux direct proportionality to the square of the concentration is in agreement with experimental curves he analyzed.

The work of Saffman [23], Drew and other leading scientists in that field was concerned with the effects of the introduction of dust particles into a clear flow of fluid. Our model is the other way around in the sense that the dust particles are subject to forces the fluid's turbulent motion is generating. The model studied by Saffman did not include interactions between dust particles which is not physically realistic. Our model, however, does include these interactions as well as other interactions since we used the propagator theory to get a very general description of the above situation.

V. I. Palaniswamy and C. M. Purushotham [21] use a variable $\tau = m/s$ called the relaxation time which is a measure of time taken by the fine dust particles to adjust to the local fluid motion. Here m is the mass of dust particle and s is the Stokes' drag coefficient. The relaxation time is assumed to be very much less than the time characterizing the basic flow. Although this is not a bad assumption, we would still like to have a theory that makes the least number of approximations in its general and initial formalism. Our propagator theory of fine dust avoids the above approximation because it treats streamlines and fine dust particles as mathematical points with different densities. That is, there is no difference in the pattern of the variation of spatial coordinate of the tip of streamlines and the dust particles with respect to time. In their paper they introduced two separate equations which correspond to the fluid motion and the dust in the fluid. These equations have only one coupling factor which corresponds to the viscous drag force. Besides being too simple in formulating the interactions between the particles and the streamlines, they did not include the rotational flow which we have a plenty of during a turbulent flow. This is due to the fact that they only considered laminar flow with small disturbances. Our introduction of a vector potential allowed rotational flow and that in turn motivated the introduction of a drag force and drag coefficients as components of a matrix. We let this drag matrix be a function of position and time and considered its spatial variation to reflect the changes in the properties of the fluid as it flows randomly. Also, in their paper, they introduce particle motion in a fluid equation which includes the particles' velocity. We can take the particles' velocity in their equation to be the negative gradient of the new scalar potential Φ we had introduced above. By substituting this expression for the particles' velocity into Palaniswamy and Purushotham's equation of particle motion, we have included all the possible interactions that may occur. This demonstrates the application of our propagator theory to other situations.

Our formalism of active forces in terms of a Lagrangian density allows us to include different forces such as the drag force as well as the lifting forces on the particulate, a force also considered in a series of papers by Drew [9]. As Drew's paper gave a term proportional to the lift force, the term he gave have no information about the origin of the lift force. It is more of a descriptive expression than physical. However, we can study the origin of the lift force more deeply by our model since lift force can be generated by rotational flow and we developed a formalism particularly for rotational flow. This lift force which occurs in our formalism is very important. Otterman and Lee [20], Dussan and Lee [10] have used continuum models of lift force in some situations where lift force was considered to be important.

In order to give a coherent description of the flows in the near shore region, we then take up the study of water at the surface. Chapter 4 of our paper concerns matter forces in a one dimensional, surface liquid water, specifically, the surface tension of water. As yet, there have been no successful explanations of the nature of the forces involved. The only mathematical model for describing surface tension is empirically based rather than being derived from first principles. The model introduced here tries to describe the forces involved on the surface of the water causing the surface to behave like a coherent sheet of matter. The model given here succeeds in carrying out this program from a fundamental point of view.

In this chapter we develop the equations that describe the surface tension for water. We give two mathematical models pertaining to the same physical model. The first mathematical model considers only the nearest neighbor interaction and the second mathematical model considers all the dipoles surrounding a specific one. More precisely, we are stating that the origin of the forces of surface tension are due to the electromagnetic, van der Waals forces that exist between any two permanent dipoles. The surface tension, we argue, will be due to the arrangement of these van der Waals dipoles. The arrangement is important in the theory because the van der Waals potential between two dipoles depends on the orientation of the two dipoles. The average over all the angles (orientations) a dipole can make with the line that joins the center of the dipole with the center of another dipole next to it gives the potential energy of the van der Waals forces between them a factor of $(-3/2)$. This would be the case of a van der Waals gas. Since we are dealing with a van der Waals liquid, we choose the parallel arrangement of dipoles on the surface of the liquid, causing forces between the dipoles on the surface to be attractive. This highly ordered arrangement implies minimum kinetic energy. As the kinetic energy is increased, the arrangement becomes more disordered and random. This randomness makes anti-parallel arrangements possible, which are really repellent dipoles on the two-dimensional (one-dimensional) surface. The proliferation of these repellent points on the surface as the temperature is increased breaks the coherency of the surface sheet. This, we believe, is an excellent interpretation of what happens on the surface during boiling or evaporation processes where the breakage of the surface sheet allows volume molecules to escape. Alternatively, we can break the surface sheet by setting the volume molecules into a turbulent mode which will, in turn, cause the surface dipoles to become randomly oriented and this will lead to a decrease in the surface tension. This argument is supported by J. M. Floryan, S. H. Davis and R. E. Kelly [12] in their study of instabilities of a liquid film flowing down a slightly inclined plane. They defined a Reynolds' number and an equation for surface tension number given, respectively, by

$$R = \frac{1}{2}\rho^2 g h \mu^{-2} \sin\beta$$
$$\xi = (3\rho T^3/g\mu^4)^{1/3}$$

where these variables, together with their dimensions, are

μ = kinematic viscosity, (L^2/T)

ρ = density, M/L^3

T = surface tension, M/T^2

g = $g^\star \sin\beta$, (L/T^2)

$g^\star$ = acceleration of gravity, (L/T^2)

β = angle of inclination from horizontal
($\beta = \pi/2$ for vertical films)

L = line of wave inception, (L)

h = thickness of flowing film

Here L refers to length, T to time, and M to mass. The equation for ξ was found by Anshus [2]. If we solve for μ^2 in the Anshus equation and substitute it into the equation for the Reynolds' number, we obtain $R = \rho^2 g h^3 g^{1/2} \xi^{3/2} \sin\beta / 2T^{3/2}(3\rho)^{1/2}$. The resulting equation implies that the Reynolds' number and surface tension are inversely proportional. It is stated in that paper that as R is increased from zero, the state of the system changes from infinitesimally stable to unstable (turbulent). Therefore, if we are to increase the instability or turbulence of the system's state by increasing the Reynolds' number, we will be decreasing the surface tension. This is analogous to what happens to the surface dipoles by increasing the temperature. In both cases a random arrangement of the surface dipoles (a decrease in the surface tension) was reached. In short, we are saying that the physical implications of the Reynolds' number's being inversely proportional to the surface tension is compatible with our parallel dipole arrangement model. Therefore, we have a good example of the comparability of our dipole arrangement model with equations previously found concerning the same phenomenon.

The phenomenon of a spherical water liquid drop can also be interpreted in terms of the dipole arrangement model. If the surface of a liquid drop of water has parallel dipoles in one direction and the layer right beneath it has parallel dipoles in exactly the opposite direction, then the resulting force between the layers is attractive. Since water is isotropic, the attraction between the layers will cause an equal inward force on the drop from all directions causing it to have a spherical shape.

We showed above a number of different phenomena that we were able to interpret in terms of our physical model of parallel dipoles which we associate with the surface tension. In terms of mathematical modeling, the equations we present below are more successful in interpreting the natural phenomenon of surface tension than previous attempts and these equations include the familiar empirical ones. The second model equation that we found, which describes the surface tension, has an inverse temperature dependence. This inverse dependence was somehow miraculous since our second equation of surface tension was multiplied by a factor of $k_B T^2$ where k_B = Boltzmann constant, T = temperature. We not only obtained a correct temperature dependence on the right hand side of the surface tension equation but we also obtained a term which we called the "bond strength" term and which contained that inverse temperature dependence. We also obtained a modulating factor in both equations dependent on the angle the surface makes with the equilibrium surface. The final result of the second model was expressed in terms of a general potential with an extra term for which there was no physical interpretation. If we do not assume the forces between the molecules on the surface of the water to be given in terms of a certain potential then the final result is an equation in terms of a general radial potential together with an extra term we would normally expect to vanish.When we chose the potential to be of a van der Waals type, we found the extra term to converge to zero very rapidly. So we can see that our second equation of surface tension has a variety of advantages, including the ability to predict the correct form of a potential (as well as the specific arrangement of van der Waals dipoles) that exists between surface water molecules.

Saleh Tanveer [28] gives an equation for surface tension which, after some manipulations, we can write as $T = \partial P/\partial n/\partial(1/R)/\partial n$, where P is the pressure, R is the curvature and n is the normal to the surface. Since this result is global, it does not show the origin of the forces or any terms relating to the bonds between the water molecules. In comparison with our result, the above equation for surface tension is the modulating factor times the applied Newtonian force. The microscopic bond strength term is zero.

Anshus [2] gives us an equation for the surface tension number ξ. The equation is $\xi = (3\rho T^3/g\mu^4)^{1/3}$, where ρ is the density of water, T is the surface tension, g is the gravitational constant and μ is the viscosity. His equation shows the surface tension in relation to other parameters, but it also does not show the origin of the surface tension. The equation, however, does have the advantage of giving us information about the numerical accuracy of our equations of surface tension. Since we have a numerical value for ξ and the other parameters are known in the above equation, we can then solve for the surface tension. Conversely, we can substitute numerical values for the surface tension predicted by our theoretical models into the Anshus equation and compare the accuracy of the resulting ξ's with the calculated one by Anshus. By doing that we are also choosing the best of the two mathematical models introduced to approximate the surface tension phenomenon.

One of the situations in which our model of surface tension becomes important is when we study boundary layer flow over a compliant surface or membrane. Research on the interaction between boundary layer flows and compliant surfaces was first motivated by experimental reports on drag-reducing capabilities of such coatings. A simple example of a compliant surface is a one-dimensional spring backed membrane governed by the equation

$$\frac{\partial \eta}{\partial t} = \frac{T}{m^\star}\frac{\partial^2 \eta}{\partial x^2} - d\frac{\partial \eta}{\partial t} - \frac{\kappa}{m^\star}\eta + f$$

where T is the surface tension, η is the vertical displacement and $m^\star$ is the superficial density, that is, mass per unit area. The above equation was widely used by Domaradzki and Metcalfe [8] to identify basic properties of a surface that could delay flow in laminar boundary layers.

Finally, we demonstrate the application of chapters 3 and 4 in chapter 5. It is an experimental fact that the interaction and dust scattering by streamlines occur at certain critical, dynamical values. In chapter 5 we use the force equation we found in chapter 4 between water molecules to get an expression for the critical value of the velocity of the scattering streamline that triggers the scattering. Chapter 3 provides us with the mathematical expression for the streamline and the physical picture associated with it. We arrive at an interesting result which is a cubic equation for the critical velocity with the matter force on its right hand side. We only consider forces due to van der Waals attraction which implies noncohesive sediments. For cohesive sediments, however, we can add a force term due to the coulombic attraction between the particles in the sediment. An interesting inverse relation between the critical velocity and the temperature is also obtained here.

2 Summary Of Results

To give a clear picture of our theory we introduce the following equations and results. We will write down the equations and tell what they are supposed to describe instead of going into the details of defining every variable. Later, in the theoretical modeling part, we will derive or define every variable.

As in the case of chapter 3 we introduce the components of the equation of motion of scattered dust by a single water beam (a continuous collection of streamlines) in spherical coordinates. Therefore, the radial, the polar, and the azimuthal equations are, respectively, given by

$$\begin{aligned}
&\int d^3x\,\rho(\vec{x},t)\frac{\partial v_r(\vec{x},t)}{\partial t} - \int d^3x\,\rho(\vec{x},t)[v_\theta(\vec{x},t)]^2 - \int d^3x\,\rho(\vec{x},t)[v_\varphi(\vec{x},t)]^2\sin[\theta(\vec{x},t)] \\
&\quad + \int d^3x\,\rho(\vec{x},t)v_r(\vec{x},t)\frac{\partial v_r(\vec{x},t)}{\partial r} + \int d^3x\,\rho(\vec{x},t)\frac{v_\theta(\vec{x},t)}{r}\left[\frac{\partial v_r(\vec{x},t)}{\partial\theta} - v_\theta(\vec{x},t)\right] \\
&\quad + \int d^3x\,\rho(\vec{x},t)\frac{v_\varphi(\vec{x},t)}{r\sin\theta}\left[\frac{\partial v_r(\vec{x},t)}{\partial\varphi} - v_\varphi(\vec{x},t)\sin[\theta(\vec{x},t)]\right] \\
&\quad + \int d^3x\,g\rho(\vec{x},t)\cos[\theta(\vec{x},t)] - \int d^3x\,\rho(\vec{x},t)r(\vec{x},t)v_\theta^2(\vec{x},t) \\
&\quad - \int d^3x\,\rho(\vec{x},t)r(\vec{x},t)\sin[\theta(\vec{x},t)]v_\varphi^2(\vec{x},t) = \sum_{i=1}^{2} F_r^i(\vec{x},t),
\end{aligned}$$

$$\begin{aligned}
&\int d^3x\,\rho(\vec{x},t)r^2(\vec{x},t)\frac{\partial v_\theta(\vec{x},t)}{\partial t} \\
&\quad + \int d^3x\rho(\vec{x},t)r^2(\vec{x},t)v_r(\vec{x},t)v_\theta(\vec{x},t) + \int d^3x\rho(\vec{x},t)r^2(\vec{x},t)v_r(\vec{x},t)\frac{\partial v_\theta(\vec{x},t)}{\partial r} \\
&\quad + 3\int d^3x\rho(\vec{x},t)r(\vec{x},t)v_\theta(\vec{x},t)v_r(\vec{x},t) + \int d^3x\rho(\vec{x},t)r(\vec{x},t)v_\theta(\vec{x},t)\frac{\partial v_\theta(\vec{x},t)}{\partial\theta} \\
&\quad + \int d^3x\rho(\vec{x},t)r(\vec{x},t)\frac{v_\varphi(\vec{x},t)}{\sin\theta}\frac{\partial v_\theta(\vec{x},t)}{\partial\varphi} + \int d^3x\rho(\vec{x},t)r(\vec{x},t)[v_\varphi(\vec{x},t)]^2(\cos\theta - \cot\theta) \\
&\quad - \int d^3x\rho(\vec{x},t)gr(\vec{x},t)\sin\theta = \sum_{i=1}^{2} F_\theta^i(\vec{x},t),
\end{aligned}$$

and

$$\int d^3x\, \rho(\vec{x},t) r^2(\vec{x},t) \sin[\theta(\vec{x},t)] \left\{ \frac{\partial v_\varphi(\vec{x},t)}{\partial t} + v_r(\vec{x},t) v_\varphi(\vec{x},t) \sin[\theta(\vec{x},t)] \right.$$
$$+ v_\theta(\vec{x},t) v_\varphi(\vec{x},t) \cos[\theta(\vec{x},t)] + v_r(\vec{x},t) \frac{\partial v_\varphi(\vec{x},t)}{\partial r} + \frac{v_\theta(\vec{x},t)}{r} \frac{\partial v_\varphi(\vec{x},t)}{\partial \theta}$$
$$\left. + \frac{v_\varphi(\vec{x},t)}{r \sin\theta} \left[v_r(\vec{x},t) \sin\theta + v_\theta(\vec{x},t) \cos\theta + \frac{\partial v_\varphi(\vec{x},t)}{\partial \varphi} \right] \right\}$$
$$+ \int d^3x\, \rho(\vec{x},t) r^2(\vec{x},t) \cos[\theta(\vec{x},t)] v_\theta(\vec{x},t) v_\varphi(\vec{x},t)$$
$$+ 2 \int d^3x\, \rho(\vec{x},t) r(\vec{x},t) v_r(\vec{x},t) \sin[\theta(\vec{x},t)] v_\varphi(\vec{x},t) = \sum_{i=1}^{2} F^i_\varphi(\vec{x},t).$$

Where $\rho(\vec{x},t)$ is the particle mass density and where $r(\vec{x},t)$ is the radial component, $\theta(\vec{x},t)$ is the polar component, and $\varphi(\vec{x},t)$ is the azimuthal component. In all the expressions above $\vec{x}$ is the position vector and t is the time. Accordingly we define $\partial r/\partial t \equiv v_r$, $\partial\theta/\partial t \equiv v_\theta$, and $\partial\varphi/\partial t \equiv v_\varphi$. The components of $\vec{F}^{(1)}(\vec{x})$ and $\vec{F}^{(2)}(\vec{x})$ that appear on the right hand side of each equation are the forces involved in the scattering and the damping of the dust particles. The equations for the multi beam situation are given by equations (3.4.130), (3.4.131), and (3.4.132), which are too lengthy to reproduce here. $\vec{F}^{(1)}(\vec{x})$ and $\vec{F}^{(2)}(\vec{x})$ in the multi beam picture are given by

$$\Big(F_r^{1(\zeta)}(\vec{x},t), F_\theta^{1(\zeta)}(\vec{x},t), F_\varphi^{1(\zeta)}(\vec{x},t)\Big) \equiv$$
$$\int_{x_i(\zeta)}^{x_f(\zeta)} dx(\zeta) \int_{y_i(\zeta)}^{y_f(\zeta)} dy(\zeta) \Big(\cos\theta\big(x(\zeta),y(\zeta),t\big), -r\big(x(\zeta),y(\zeta),t\big)\sin\theta\big(x(\zeta),y(\zeta),t\big), 0\Big)$$
$$\times \int_{-\infty}^{\infty} d\tau\, \Big\{[\eta(\vec{\nabla} v_n) + p\hat{n} + \rho\vec{v}(\vec{v}\cdot\hat{n})]\cdot\hat{e}_3\Big\}\,(x(\zeta),y(\zeta))\delta(\tau - t(\zeta))$$

and

$$\Big(F_r^{2(\zeta)}(\vec{x},t), F_\theta^{2(\zeta)}(\vec{x},t), F_\varphi^{2(\zeta)}(\vec{x},t)\Big) \equiv$$

$$\int_S \int_0^{z_0} dz(\zeta) \int_{-\infty}^{\infty} dx(\zeta) \Big(\varpi_r \Big(r\big(x(\zeta), z(\zeta), t\big), \theta\big(x(\zeta), z(\zeta), t\big), \varphi\big(x(\zeta), z(\zeta), t\big)\Big),$$
$$\varpi_\theta \Big(r\big(x(\zeta), z(\zeta), t\big), \theta\big(x(\zeta), z(\zeta), t\big), \varphi\big(x(\zeta), z(\zeta), t\big)\Big),$$
$$\varpi_\varphi \Big(r\big(x(\zeta), z(\zeta), t\big), \theta\big(x(\zeta), z(\zeta), t\big), \varphi\big(x(\zeta), z(\zeta), t\big)\Big)\Big)$$
$$\times \int_{t(\zeta)}^{\infty} d\tau da(x(\zeta), z(\zeta)) \int_{-\pi/2}^{\pi/2} d\vartheta(\vec{x}(\zeta), z(\zeta)) v^{inc}(x(\zeta), z(\zeta)) \cos[\vartheta(x(\zeta), z(\zeta))]$$
$$\times \Big[\Big|[\vec{\Pi}^c \cdot \hat{y}'](x(\zeta), z(\zeta))\Big| + \alpha \Big|[\vec{\Pi}^{c'} \cdot \hat{y}'](x(\zeta), z(\zeta))\Big| \Big]$$

The quantity $|\vec{\Pi}^c(y(\zeta), z(\zeta)) \cdot \hat{y}'|$ that occurs in the integrand of $\vec{F}^{(2)}(\vec{x})$ is the canonical momentum density. The primed quantity is the reflected one. So we write out the expression for the incident canonical momentum density and the primed one would then be similar in form. So we have

$$\begin{aligned} \Big|\vec{\Pi}^c(y(\zeta), z(\zeta)) \cdot \hat{y}'\Big| &= \Big| \sum_{j=1}^{3} \Big\{ \Big[a_j + \rho \dot{A}_j(y(\zeta), z(\zeta)) - \rho \alpha_{jj} A_j(y(\zeta), z(\zeta)) \\ &\quad - \lambda(\hat{t} \cdot \vec{\nabla} \alpha_{jj}) A_j(y(\zeta), z(\zeta)) + b_j \dot{\Phi}(y(\zeta), z(\zeta)) \Big] \Big\} (\hat{e}_j \cdot \hat{y}') \Big| \end{aligned}$$

which we obtain by taking the partials of the Lagrangian density

$$\begin{aligned} \mathscr{L} &= \Big[\dot{\phi} + \frac{1}{2}(\vec{\nabla}\phi)^2 + gz + \frac{1}{2}\rho(\dot{A}_j)^2 \\ &\quad - \frac{1}{2}\eta(\vec{\nabla} A_j)^2 - \rho \sum_{n=1}^{3} \alpha_{jn} \dot{A}_n A_j \\ &\quad - \lambda \sum_{n=1}^{3} (\hat{t} \cdot \vec{\nabla} \alpha_{jn}) \dot{A}_n A_j + \frac{1}{2}(\dot{\Phi})^2 \Big] \end{aligned}$$

with respect to the corresponding conjugate coordinate. The A_j's are components of the vector potential $\vec{A}$ we talked about previously. The equations of motion for

$\vec{A}$ are

$$\rho\frac{\partial^2 A_j}{\partial t^2} - \eta\nabla^2 A_j = \rho\sum_{n=1}^{3}\dot{\alpha}_{jn}A_n + \lambda\sum_{n=1}^{3}\sum_{\ell=1}^{2}t_\ell\frac{\partial\dot{\alpha}_{jn}}{\partial x_\ell}A_n \qquad j = 1,2,3$$

with the right hand side being our proposed force term. The quantity ϕ in $\mathscr{L}$ above is the solution to the Laplace equation.

The final thing we mention about the above $\mathscr{L}$ is that the last term appears as $\frac{1}{2}(\dot{\Phi})^2$. The negative gradient of Φ gives us the diffused particles velocity. We invented Φ in particular to be able to use the propagator theory in formulating interactions that are due to unavoidable interferences of different beams in the multi beam situation. The propagator theory is applied in the following way. Since

$$\dot{\Phi}(\vec{x},t) - D\nabla^2\Phi(\vec{x},t) = \kappa 2\vec{v}\cdot\vec{v}_p\Phi(\vec{x},t)$$

is the equation we found for Φ, where D is a constant, we expect the propagator to be included through $\Phi(\vec{x}(\zeta),t)$ shown above, and it is. The equation of motion for $\Phi(\vec{x}',t')$ (single beam at position $\vec{x}'$ and time t') we found to be

$$\frac{\partial\Phi(\vec{x}',t')}{\partial t'} = D\nabla^2\Phi(\vec{x}',t') + V(\vec{x}',t')\Phi(\vec{x}',t')$$

with $V(\vec{x},t) = (2\kappa/\rho)\Delta KE$ as the change in kinetic energy per mass density times the constant κ. The above equation has the solution

$$\Phi(\vec{x}',t') = \int d^3x G(\vec{x}',t';\vec{x},t)\psi(\vec{x},t)$$

where

$$G(\vec{x}',t';\vec{x},t) \;=\; G_0(\vec{x}',t';\vec{x},t)$$

$$+\sum_i \int d^3x_i \Delta t_i G_0(\vec{x}', t'; \vec{x}_i, t_i) V(\vec{x}_i, t_i) G_0(\vec{x}_i, t_i; \vec{x}, t)$$
$$+\sum_{\substack{i>j \\ (t_i>t_j)}} \int d^3x_i \Delta t_i d^3x_j \Delta t_j G_0(\vec{x}', t'; \vec{x}_i, t_i) V(\vec{x}_i, t_i)$$
$$\times G_0(\vec{x}_i, t_i; \vec{x}_j, t_j) V(\vec{x}_j, t_j) G_0(\vec{x}_j, t_j; \vec{x}, t) + \cdots$$

is the propagator for n interactions. This concludes chapter 3 and summarizes the important equations in that chapter.

Chapter 4 on the other hand is not as complicated as chapter 3. That should be expected since the physical situation encountered in this chapter is simpler but the phenomenon is much more mysterious. In this chapter we introduce two equations that model the surface tension of water. The first equation we derive is the nearest neighbor interaction. In this equation the surface tension of a single site of a chain of dipoles is approximated by the sites interaction with its single adjacent neighbor. This equation is

$$\tau = \frac{12\alpha}{\cos\theta} \frac{\mu^2}{r^7}$$

with θ being the angle made with the equilibrium surface, α being the polarizability, μ being the dipole moment of a water molecule and r the intermolecular distance. The second model or equation we introduce in chapter 4 is one which is derived from the principles of statistical mechanics and avoids undesirable approximations such as the one mentioned above. This equation is

$$\tau = \frac{1}{\cos\theta} \left[\left\langle \frac{dV}{dr} \right\rangle - \frac{d}{dr} \chi_b \right]$$

where θ is given as before and

$$\chi_b = \beta \sum_{ij} \langle V(r_{ij}) V(r) \rangle - \langle V(r_{ij}) \rangle \langle V(r) \rangle$$

is the bond strength term. Note the Newtonian action reaction principle involved in the equation for τ.

In the fifth and the final modelling chapter we show the application of chapter 4 with a streamline picture similar to that in chapter 3. Our result in chapter 5 is basically the following equation

$$v_{nc}^3 + \frac{2(P + \varepsilon\rho)}{\rho} v_{nc} = -\frac{8N}{\rho} \int_0^L dx \int_0^L dy \frac{\alpha\mu^2}{[r(x,y)]^6}$$

where the v_{nc} on the left hand side is the critical velocity of the streamline with direction opposite to the normal to the ground. The right hand side is then the force that "glues" the sediments on the ground together. This equation was necessary because it is experimentally observed that dust scattering by streamlines occur at certain critical values. Here we demonstrate how velocity can be one of these critical values.

3 Hydrodynamic Mass Transport

3.1 Hydrodynamic forces and rotational flow

To be able to describe the motion of a system of particles, we must introduce a coordinate system so that we may assign coordinates to those particles. Then we must know the equations of motion which the particles obey. Finally, we must have a solution to the equations of motion, typically obtained as the limit of a convergent sequence, which describes the evolution of the particles over time. Here we will use spherical coordinates in the description of the motion of the particles. These particles are dust particles initially residing on a ground that is being struck by water waves that excite the dust particles (or just particles) and set them into motion. First we think of the motion of water as being made up of streamlines in a two dimensional plane, and later we will include the third dimension. Therefore, the water particles that hit the ground follow two dimensional curved lines, or streamlines, transporting energy and momentum. The following picture demonstrates the idea.

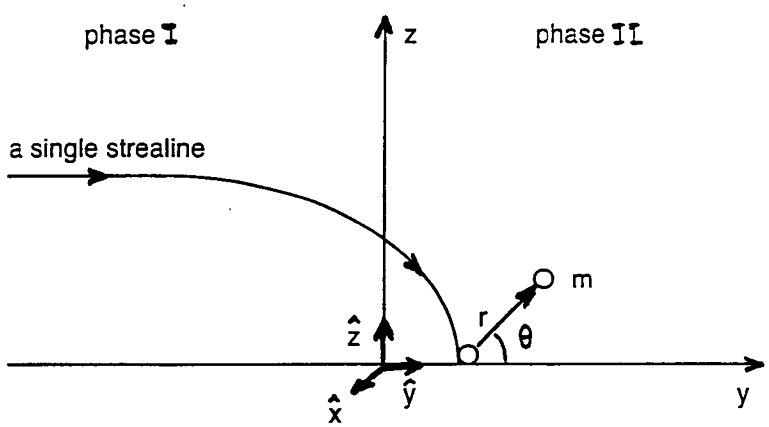

Figure 1: Showing a simple streamline hitting a single particle, and generating coordinates for the particle as it flies apart from the ground.

In Figure (1) we show a single streamline; however, internal water waves are made up of a large collection of these streamlines. We observe also that the particles will possibly be set in motion after they have been struck by the streamline. Hence, we define a polar coordinate θ_i and a radial coordinate r_i for the two dimensional ith particle with mass m_i. We take the z-axis as shown to be the axis perpendicular to the ground. Also $\vec{g} = -g\hat{z}$ in Figure (1) is the gravitational acceleration, which we take to be constant. Finally, note that Figure (1) is divided into two phases. Phase I is where the particles are at rest on the ground and phase II is where the particles have nonzero planar coordinates and are interacting with their surroundings. Now we will start the mathematical analysis of the situation we have set up. The force on the ith particle due to gravity is

$$\vec{F_i} = -m_i g\hat{z} \tag{3.1.1}$$

where from now on the subscript i refers to the ith particle, unless otherwise specified. Let $z_i = r_i \cos\theta_i$ and $y_i = r_i \sin\theta_i$. The kinetic energy of the ith particle, T_i, in terms of its polar coordinates is $T_i = \frac{1}{2}m_i(\dot{r}_i^2 + r_i^2\dot{\theta}_i^2)$. From Symon [27] the generalized equation of motion is

$$\frac{d}{dt}\frac{\partial T_i}{\partial \dot{q}_\sigma} - \frac{\partial T_i}{\partial q_\sigma} = Q^i_\sigma, \qquad \sigma = 1, 2, \tag{3.1.2}$$

where q_σ stands for generalized coordinates, with $q_1 = r$ and $q_2 = \theta$ in our case. The Q^i_σ corresponds to the generalized force acting on the ith particle. From the same source we have

$$Q^i_\sigma = f_{y_i}\frac{\partial y_i}{\partial q_\sigma} + f_{z_i}\frac{\partial z_i}{\partial q_\sigma}, \tag{3.1.3}$$

where $\vec{f} = f_y\hat{y} + f_z\hat{z}$ is the force vector which contains both the conservative and non-conservative parts. Remember that z_i and y_i are the components along the unit vectors defined in Figure (1). The radial equation, according to equation (3.1.2), of the ith particle is then

$$\frac{d}{dt}\frac{\partial}{\partial \dot{r}_i}\frac{1}{2}m_i\left[(\dot{r}_i^2 + r_i^2\dot{\theta}_i^2)\right] - \frac{\partial}{\partial r_i}\left[\frac{1}{2}m_i(\dot{r}_i^2 + r_i^2\dot{\theta}_i^2)\right] = -m_i g\frac{\partial z_i}{\partial r_i} + f_{r_i}^a + f_{r_i}^b,$$

where $dr_i/dt = \dot{r}_i$. By taking the derivatives we get

$$\frac{d}{dt}(m_i\dot{r}_i) - m_i r_i\dot{\theta}_i^2 = -m_i g\cos\theta_i + f_{r_i}^a + f_{r_i}^b. \tag{3.1.4}$$

Here we added two force terms $f_{r_i}^a$ and $f_{r_i}^b$ in addition to the conservative force shown in the first term of the right hand side of (3.1.4). The force terms $f_{r_i}^a$ and $f_{\theta_i}^a$ represent the radial and the polar components of the force delivered to the ith particle instantaneously at $t = t_0$ by the ith streamline, respectively. On the other hand, the force terms $f_{r_i}^b$ and $f_{\theta_i}^b$ represent, respectively, the radial and the polar components of forces acting in phase II that will cause the damping of the excited particles, hence govern their motion.

Similarly the polar equation of the ith particle is

$$\frac{d}{dt}\frac{\partial}{\partial \dot{\theta}_i}\left[\frac{1}{2}m_i(\dot{r}_i^2 + r_i^2\dot{\theta}_i^2)\right] - \frac{\partial}{\partial \theta_i}\left[\frac{1}{2}m_i(\dot{r}_i^2 + r_i^2\dot{\theta}_i^2)\right] = -m_i g\frac{\partial z_i}{\partial \theta_i} + f_{\theta_i}^a + f_{\theta_i}^b,$$

or

$$\frac{d}{dt}(m_i r_i^2\dot{\theta}_i) = m_i g r_i\sin\theta_i + f_{\theta_i}^a + f_{\theta_i}^b. \tag{3.1.5}$$

Therefore, we have two coupled differential equations which we rewrite as

$$m_i\ddot{r}_i + m_i g\cos\theta_i - m_i r_i\dot{\theta}_i^2 = f_{r_i}^a + f_{r_i}^b, \tag{3.1.6}$$

$$m_i r_i^2 \ddot{\theta}_i - m_i g r_i \sin\theta_i + 2 m_i r_i \dot{r}_i \dot{\theta}_i = f^a_{\theta_i} + f^b_{\theta_i}. \tag{3.1.7}$$

Our task now remains to find expressions for the two force vectors $\vec{f}_i^a$ and $\vec{f}_i^b$. If we do that, we will almost be done with our theory of the hydrodynamics of the mass transport. The only thing that will be left is to take equations (3.1.6) and (3.1.7) to a continuous limit, and as we shall see, that will generate problems of its own. Now let Π_i and Π_i' be the norms of the ith streamline momentum density before and after its collision with the ith particle, respectively. Let the norms of the ith particle momentum density before and after collision be denoted by P_i and P_i', respectively.

Now let

$$f_i^a = \int_{-\infty}^{\infty} dt\, \delta(t - t_0) \left[\frac{\partial P_i(\vec{x}, t)}{\partial t}\right] \tag{3.1.8}$$

be the force transferred to the ith particle at $t = t_0$ due to the ith streamline, where $\delta(t - t_0)$ is the Dirac delta function, and where $f^a_{r_i} = \cos\theta_i f_i^a$ and $f^a_{\theta_i} = -r_i \sin\theta_i f_i^a$ with $\vec{f}_i^a = (f^a_{y_i}, f^a_{z_i})$. From conservation of momentum we have

$$\Pi_i + P_i = \Pi_i' + P_i' \tag{3.1.9}$$

or $P_i - P_i' = \Pi_i' - \Pi_i$. Define $\Delta\Pi_i = \Pi_i - \Pi_i'$ and $\Delta P_i = P_i - P_i'$ then from (3.1.9) we have $-\Delta P_i = \Delta\Pi_i$ or

$$-\frac{\Delta P_i}{\Delta t}\Delta t = \frac{\Delta \Pi_i}{\Delta t}\Delta t$$

Assuming instantaneous collision we get

$$-\frac{\partial P_i}{\partial t}dt = \frac{\partial \Pi_i}{\partial t}dt \tag{3.1.10}$$

If we make the following redefinitions

$$\hat{x} = \hat{e}_1,\ \hat{y} = \hat{e}_2,\ \hat{z} = \hat{e}_3$$

and have flow only in the negative $\hat{e}_3$ direction of the streamline, we can write $\Pi_i = -(\vec{\Pi} \cdot \hat{e}_3)_i$. Likewise, a flow in the jth direction leads us to write $v_j = \vec{v} \cdot \hat{e}_j$. Let ρ be the fluid density and $\rho\vec{v}$ the mass flux density. Then according to Newton's second law of motion we have $\partial(\rho\vec{v})/\partial t$ as force per unit volume, which is equal to the time rate of change of streamline momentum. Let $[\partial(\rho\vec{v}/\partial t)]_i$ be the force per unit volume corresponding to the ith streamline's momentum change. Therefore, with $j = 3$ we have

$$\left[\frac{\partial(\rho v_3)}{\partial t}\right]_i = \frac{\partial \Pi_i}{\partial t}, \tag{3.1.11}$$

where at the instant of collision, that is at $t = t_0$, the only flow direction of the streamline is the flow opposite the normal to the ground that is being struck by the streamline. From Landau and Lifshitz [19], with the summation convention holding, we have

$$\left[\frac{\partial(\rho v_3)}{\partial t}\right]_i = -\left[\frac{\partial \Theta_{3k}}{\partial x_k}\right]_i$$

where $\Theta_{jk} = p\delta_{jk} + \rho v_j v_k - \sigma'_{jk}$. Here p is the pressure, δ_{jk} is the Kronecker delta, $-\sigma'_{jk}$ is the term responsible for the irreversible viscous transfer of momentum in the fluid which we chose to be zero, and Θ_{jk} is the momentum flux density tensor. From the same source we have

$$\left[\frac{\partial(\rho v_j)}{\partial t} + \rho v_k \frac{\partial v_j}{\partial x_k}\right]_i = \left[-\frac{\partial p}{\partial x_j} + \frac{\partial}{\partial x_k}\eta\left(\frac{\partial v_j}{\partial x_k} + \frac{\partial v_k}{\partial x_j}\right)\right]_i$$

which is the Navier-Stokes' equation with η as the coefficient of viscosity which in our investigation we assumed to be constant. For an incompressible fluid, we have

$\vec{\nabla} \cdot \vec{v} = 0$ or with the summation convention $\partial v_k / \partial x_k = 0$. Using the fact that

$$\frac{\partial}{\partial x_k}\frac{\partial v_k}{\partial x_j} = \frac{\partial}{\partial x_j}\frac{\partial v_k}{\partial x_k} = 0,$$

the above equation becomes

$$\left[\frac{\partial(\rho v_j)}{\partial t} + \rho v_k \frac{\partial v_j}{\partial x_k}\right]_i = \left[-\frac{\partial p}{\partial x_j} + \eta \frac{\partial^2 v_j}{\partial x_k^2}\right]_i,$$

or

$$\begin{aligned}
\left[\frac{\partial(\rho v_j)}{\partial t}\right]_i &= \left[-\left(\frac{\partial p}{\partial x_k}\delta_{jk} + \rho v_k \frac{\partial v_j}{\partial x_k} + \rho v_j \frac{\partial v_k}{\partial x_k}\right) + \eta \frac{\partial^2 v_j}{\partial x_k^2}\right]_i \\
&= \left\{-\left[\frac{\partial p}{\partial x_k}\delta_{jk} + \frac{\partial}{\partial x_k}(\rho v_j v_k)\right] + \eta \frac{\partial^2 v_j}{\partial x_k^2}\right\}_i \\
&= \left[-\frac{\partial}{\partial x_k}\left(p\delta_{jk} + \rho v_j v_k\right) + \eta \frac{\partial^2 v_j}{\partial x_k^2}\right]_i \\
&= \left[-\frac{\partial}{\partial x_k}\Theta_{jk} + \eta \frac{\partial^2 v_j}{\partial x_k^2}\right]_i .
\end{aligned}$$

Integrating the Navier-Stokes' equations over an elementary volume where the interaction occurs, we get

$$\int_{\Delta V_i}\left[\frac{\partial(\rho v_j)}{\partial t}\right]_i d^3x = -\int_{\Delta V_i}\left[\frac{\partial}{\partial x_k}\Theta_{jk}\right]_i d^3x + \int_{\Delta V_i}\left[\frac{\partial}{\partial x_k}\left\{\eta \frac{\partial v_j}{\partial x_k}\right\}\right]_i d^3x.$$

Using the Gauss divergence theorem, we find

$$\frac{\partial}{\partial t}\int_{\Delta V_i}[\rho v_j]_i \, d^3x = -\oint_{\Delta a_i}[\Theta_{jk}]_i \, da_k + \oint_{\Delta a_i}\left[\eta \frac{\partial v_j}{\partial x_k}\right]_i da_k,$$

Δa_i being the surface enclosing ΔV_i. Let us denote a unit vector normal to a surface by $\hat{n} = \cos\alpha \hat{e}_1 + \cos\beta \hat{e}_2 + \cos\gamma \hat{e}_3$. Then $\hat{n} \cdot \hat{e}_k = n_k$ is the kth direction cosine. Accordingly, $d\vec{a} \cdot \hat{e}_k = da_k = da\,\hat{n} \cdot \hat{e}_k = da\,n_k$ that is

$$da_k = n_k\,da \tag{3.1.12}$$

Using equation (3.1.12), we can rewrite the above integrated equations as

$$\begin{aligned} \frac{\partial}{\partial t} \int_{\Delta V_i} [\rho v_j]_i\, d^3x &= -\oint_{\Delta a_i} [\Theta_{jk}]_i\, n_k\, da + \oint_{\Delta a_i} \left[\eta \frac{\partial v_j}{\partial x_k}\right]_i n_k\, da \\ &= \oint_{\Delta a_i} \left[-\Theta_{jk} + \eta \left(\frac{\partial v_j}{\partial x_k}\right)\right]_i n_k\, da. \end{aligned} \tag{3.1.13}$$

From Figure (1), we have chosen the coordinates so that at $t = t_0$, $\alpha = \beta = \pi/2$, $\gamma = \pi$, so $\hat{n} = -\hat{e}_3 = -\hat{z}$, that is

$$n_k = \begin{cases} 0 & \text{if } k = 1, 2, \\ -1 & \text{if } k = 3. \end{cases}$$

The first term on the right hand side of (3.1.13) is then

$$\Theta_{jk} n_k\, da \;=\; -\Theta_{j3}\, da = -p\delta_{j3}\, da + \rho v_j v_3(-da) = \begin{cases} -(pda + \rho v_3^2 da) & \text{if } j = 3, \\ -\rho v_j v_3 da & \text{if } j = 1, 2. \end{cases}$$

At $t = t_0$, we consider for definiteness the case where $j = 3$ so the above expression becomes

$$\Theta_{jk} n_k\, da = -(p\, da + \rho v_3^2\, da) = [p\hat{n} + \rho\vec{v}(\vec{v} \cdot \hat{n})] \cdot \hat{e}_3\, da.$$

The other components can be calculated similarly. We continue to focus on the case $j = 3$. Then the second term on the right hand side of (3.1.13) is

$$\begin{aligned} \eta \left(\frac{\partial v_j}{\partial x_k}\right) n_k\, da &= \eta \left(\frac{\partial v_3}{\partial x_3}\right) n_3\, da = -\eta \frac{\partial v_3}{\partial x_3}\, da \\ &= -\eta(\vec{\nabla} v_3) \cdot \hat{e}_3\, da. \end{aligned}$$

Equation (3.1.13) is then

$$\frac{\partial}{\partial t}\int_{\Delta V_i}[\rho v_3]_i\, d^3x = -\oint_{\Delta a_i}\left\{[p\hat{n}+\rho\vec{v}(\vec{v}\cdot\hat{n})]\cdot\hat{e}_3\right\}_i da - \oint_{\Delta a_i}[\eta(\vec{\nabla}v_3)\cdot\hat{e}_3]_i\, da$$

which means that we can write

$$\left\{\frac{\partial(\rho v_3)}{\partial t}\right\}_i = -\left\{[p\hat{n}+\rho\vec{v}(\vec{v}\cdot\hat{n})+\eta(\vec{\nabla}v_3)]\cdot\hat{e}_3\right\}_i. \tag{3.1.14}$$

Replace v_3 by $-v_n$ and use equation (3.1.11) to write

$$\frac{\partial\Pi_i}{\partial t} = -\left\{[p\hat{n}+\rho\vec{v}(\vec{v}\cdot\hat{n})-\eta(\vec{\nabla}v_n)]\cdot\hat{e}_3\right\}_i.$$

From equation (3.1.10)

$$\left(\frac{\partial P_i}{\partial t}\right)dt = \left\{[-\eta(\vec{\nabla}v_n)+p\hat{n}+\rho\vec{v}(\vec{v}\cdot\hat{n})]\cdot\hat{e}_3\right\}_i dt. \tag{3.1.15}$$

Substitute equation (3.1.15) into (3.1.8) to get

$$f_i^a = \int_{-\infty}^{\infty} dt\,\delta(t-t_0)\left\{[-\eta(\vec{\nabla}v_n)+p\hat{n}+\rho\vec{v}(\vec{v}\cdot\hat{n})]\cdot\hat{e}_3\right\}_i. \tag{3.1.16}$$

Note that the viscosity exerts a damping effect on the force delivered to the particle by the streamline because of the negative sign.

The right hand side of (3.1.6) and (3.1.7) are force terms. However, (3.1.8) is a force density. Note that the same notation for f_i^a is used to denote the force density. To be able to use (3.1.16) in (3.1.6) and (3.1.7) we must write the left hand side of (3.1.6) and (3.1.7) in terms of densities. Recall that ΔV_i was defined as the interaction volume of streamline with residing particles. Therefore, we define

$\rho_i \equiv m_i/\Delta V_i$ as the mass density of the ith particle. Similarly $f^b_{r_i,\theta_i}$ are redefined as the force densities. Equations (3.1.6) and (3.1.7) with t on the right hand sides relabelled as τ and t_0 as t are then

$$\rho_i\ddot{r}_i + \rho_i g\cos\theta_i - \rho_i r_i\dot{\theta}_i^2 =$$
$$\cos\theta_i \int_{-\infty}^{\infty} d\tau\,\delta(\tau - t)\left\{[-\eta(\vec{\nabla}v_n) + p\hat{n} + \rho\vec{v}(\vec{v}\cdot\hat{n})]\cdot\hat{e}_3\right\}_i + f^b_{r_i} \quad (3.1.17)$$

$$\rho_i r_i^2\ddot{\theta}_i - \rho_i g r_i\sin\theta_i + 2\rho_i r_i\dot{r}_i\dot{\theta}_i = -r_i\sin\theta_i \int_{-\infty}^{\infty} d\tau\,\delta(\tau - t)$$
$$\times\left\{-\eta(\vec{\nabla}v_n)\cdot\hat{e}_j + [p\hat{n} + \rho\vec{v}(\vec{v}\cdot\hat{n})]\cdot\hat{e}_3\right\}_i + f^b_{\theta_i} \quad (3.1.18)$$

respectively. Here $i = 1,\ldots,N$ with N as the total number of particles (streamlines) considered and it can be as large as we want it to be. In fact we will take N large enough such that (3.1.17) and (3.1.18) can be written in continuous instead of "discretised" form. But before we do all that, we must find expressions for $f^b_{r_i,\theta_i}$. We do that next.

Consider the following two figures that were used by R. Wangsness [30].

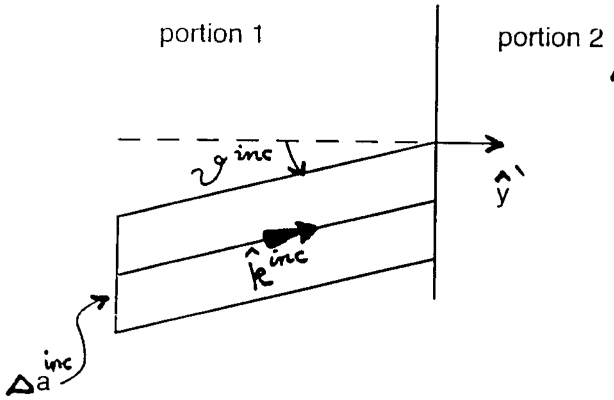

Figure 2: Showing incident wave with wavevector $\hat{k}^{inc}$, cross section Δa^{inc} and angle of incidence ϑ^{inc}. $\hat{y}'$ is directionally the same as $\hat{y}$ in Figure (1).

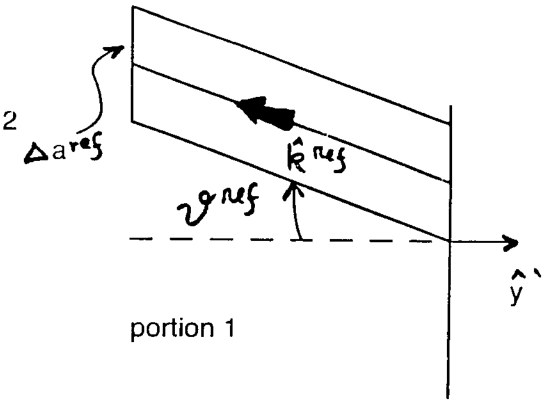

Figure 3: Showing reflected wave with wavevector $\hat{k}^{ref}$, cross section Δa^{ref} and angle of reflectance ϑ^{ref}.

We introduce the following notation and definitions

$$
\begin{aligned}
\hat{y}' &= \text{A unit vector as } \hat{y} \text{ was in Figure (1) except that it is translated} \\
& \text{in the positive z-direction.} \\
\hat{k}_i^{inc} &= \text{The incident unit wave vector corresponding to the ith streamline.} \\
\hat{k}_i^{ref} &= \text{The reflected unit wave vector corresponding to the ith streamline.} \\
v_i^{inc} &= \text{The speed of the ith incident streamline.} \\
v_i^{ref} &= \text{The speed of the ith reflected streamline.} \\
v_i^{inc}\Delta t &= \text{The distance the ith incident streamline travels in time } \Delta t. \\
v_i^{ref}\Delta t &= \text{The distance the ith reflected streamline travels in time } \Delta t. \\
G_{in}^{init} &= \text{Initial momentum density of the ith streamline normal to portion 1.} \\
G_{in}^{fin} &= \text{Final momentum density of the ith streamline normal to portion 1.}
\end{aligned}
$$

$\mathcal{G}_{in}^{init}$ = Initial momentum of the ith streamline normal to portion 1.

$\mathcal{G}_{in}^{fin}$ = Final momentum of the ith streamline normal to portion 1.

$\vec{\Pi}_i^c$ = Canonical momentum associated with the fluid velocity potential fields (we explain later what they are).

We now associate a wave vector with each traveling streamline and use Figures (2) and (3) to describe incident and reflected streamlines. An important comment here is that the plane dividing portion 1 and 2 in Figures (2) and (3) is exactly the same plane dividing phase I and phase II in Figure (1). In this manner we are able to identify the forces generated in phase II that are caused by the streamlines colliding with the plane dividing phases I and II. This dividing plane is, as we know, fictitious, so there is no reflection of streamlines by this dividing plane. However, by considering a situation like this, we will be able to obtain a mathematical expression for the forces exerted on portion 2 (phase II). Let us now go back to our model of streamlines with wave vectors defined for them. The incident wave brings up momentum to the fictitious surface dividing portions 1 and 2, while the reflected wave carries it away at a different rate. This causes a momentum change in portion 1, which from Newton's third law implies that portion 2 exerted an equal and opposite force on portion 1. Hence, we are able to get expressions for the forces on the disturbed particles, by the bombarding streamlines from phase I, in phase II. These forces in phase II on the disturbed particles are also due to the disturbed particles interacting (colliding) with themselves and other streamlines (other than the ones that disturbed them), but we will discuss that later in our interaction of interferences formalism. We claim that the momentum density of the bombarding streamlines is transported by means of the canonical momentum density associated with the fluid velocity potential fields. This directly motivates us to study these potential fields very carefully. Let us now get some expressions.

From the claim we made above, Figures (2) and (3), and the definitions we introduced we have

$$\begin{aligned} \mathcal{G}_{in}^{init} &= \left(\vec{\Pi}_i^c \cdot \hat{y}'\right)(v_i^{inc}\Delta t)\Delta a_i^{inc} \\ \mathcal{G}_{in}^{fin} &= \left(\vec{\Pi}_i^{c'} \cdot \hat{y}'\right)(v_i^{ref}\Delta t)\Delta a_i^{ref}, \end{aligned}$$

where $\vec{\Pi}_i^c$ and $\vec{\Pi}_i^{c'}$ are the canonical momentum densities associated with the fluid velocity potential fields of the incident and reflected ith streamline, respectively. The momentum change in portion 1 is

$$\mathcal{G}_{in}^{fin} - \mathcal{G}_{in}^{init} \equiv \Delta\mathcal{G}_{1n}$$

or

$$\Delta\mathcal{G}_{1n} = \left[\left(\vec{\Pi}_i^{c'} \cdot \hat{y}'\right)(v_i^{ref}\Delta t) - \left(\vec{\Pi}_i^c \cdot \hat{y}'\right)(v_i^{inc}\Delta t)\right]\Delta a_i$$

where we set $\Delta a_i^{inc} = \Delta a_i^{ref} = \Delta a_i$. The above equation can be written as

$$\Delta\mathcal{G}_{1n} = -(\Delta a_i \Delta t)\cos\vartheta_i \left[v_i^{ref}\left|\vec{\Pi}_i^{c'} \cdot \hat{y}'\right| + v_i^{inc}\left|\vec{\Pi}_i^c \cdot \hat{y}'\right|\right], \tag{3.1.19}$$

where $\vartheta_i^{inc} = \vartheta_i^{ref} = \vartheta_i$. If we take into account the change in the streamline velocity due to the friction between the water molecules, then

$$v_i^{ref} = \alpha v_i^{inc}, \tag{3.1.20}$$

where $|\alpha| \leq 1$ is an empirically determined dissipation coefficient. Equation (3.1.19) is then

$$\Delta\mathcal{G}_{1n} = -(\Delta a_i \Delta t) v_i^{inc} \cos\vartheta_i \left[\left|\vec{\Pi}_i^c \cdot \hat{y}'\right| + \alpha \left|\vec{\Pi}_i^{c'} \cdot \hat{y}'\right| \right] ; \tag{3.1.21}$$

accordingly, we have (see the definitions given above)

$$\begin{aligned} \Delta G_{1n} &= G_{in}^{fin} - G_{in}^{init} = \Delta\mathcal{G}_{1n}/(\text{unit volume}) \\ F_{1n}^i &= \vec{F}_1^i \cdot \hat{y}' = \frac{\Delta G_{1n}}{\Delta t} = \Delta\mathcal{G}_{1n}/\Delta t(\text{unit volume}) \\ &= -\Delta a_i v_i^{inc} \cos\vartheta_i \left[\left|\vec{\Pi}_i^c \cdot \hat{y}'\right| + \alpha \left|\vec{\Pi}_i^{c'} \cdot \hat{y}'\right| \right] , \end{aligned} \tag{3.1.22}$$

where F_{1n}^i is the normal force density on the ith streamline in portion 1 and the absolute values in (3.1.22) are now, using the same notation, the canonical momentum densities associated with the fluid velocity potential fields. From Newton's third law $F_{1n}^i = -F_{2n}^i$ and so

$$F_{2n}^i = \Delta a_i v_i^{inc} \cos\vartheta_i \left[\left|\vec{\Pi}_i^c \cdot \hat{y}'\right| + \alpha \left|\vec{\Pi}_i^{c'} \cdot \hat{y}'\right| \right] \tag{3.1.23}$$

is the force per unit volume applied on portion 2 through the element of area Δa_i of the plane dividing phase I and phase II. Recall that our purpose was to find an expression for the force term f_i^b, where $f_{r_i}^b = \varpi_r(r_i, \theta_i) f_i^b$, $f_{\theta_i}^b = \varpi_\theta(r_i, \theta_i) f_i^b$ with $\varpi_r(r_i, \theta_i) \equiv \sin\theta_i + \cos\theta_i$, $\varpi_\theta(r_i, \theta_i) \equiv r_i(\cos\theta_i - \sin\theta_i)$, and $\vec{f}_i^b = (f_{r_i}^b, f_{\theta_i}^b)$. Equation (3.1.23) permits us to do that and we let

$$f_i^b = \int_t^\infty d\tau\, F_{2n}^i. \tag{3.1.24}$$

Equations (3.1.17), (3.1.18), and (3.1.23) together with (3.1.24) are our solution to the problem of mass (dust particle) transport by the water. The only thing left for us to do is to augment equation (3.1.24) by finding rigorous expressions for the canonical momentum densities in equation (3.1.23).

From H. Goldstein [13] a component of the canonical momentum density is

$$\Pi_\rho(\vec{x},t) = \frac{\partial \mathscr{L}}{\partial \dot{\eta}_\rho} \qquad \rho = 1,2,3. \tag{3.1.25}$$

where $\mathscr{L}$ is the Lagrangian density with $\mathscr{L} = \mathscr{L}(\eta_\rho, \vec{\nabla}\eta_\rho, \dot{\eta}_\rho, \vec{x}, t)$, $\eta_\rho(\vec{x},t)$ is the generalized potential field from which we can get fluid velocity with the subscript indicating a specific component, and $\Pi_\rho(\vec{x},t)$ are the canonical momentum densities of the three dimensional field $\vec{\eta} = \eta_1\hat{e}_1 + \eta_2\hat{e}_2 + \eta_3\hat{e}_3$. The functional dependence of $\mathscr{L}$ shown above is the most general one. However, $\mathscr{L}$ does not have to be a function of all the variables shown above in $\mathscr{L}$.

Before dealing with the general flows, we consider briefly the case of irrotational, incompressible flows. From the incompressibility we have $\vec{\nabla} \cdot \vec{v} = 0$ and from irrotationality, $\vec{\nabla} \times \vec{v} = 0$ which implies the existence of a scalar valued function, $\phi(\vec{x},t)$, such that

$$\vec{v}(\vec{x},t) = -\vec{\nabla}\phi(\vec{x},t) \tag{3.1.26}$$

and consequently

$$\nabla^2\phi = 0. \tag{3.1.27}$$

Equation (3.1.26) defines the specific velocity potential field we were looking for. Equation (3.1.27) is the Laplace equation and so ϕ is harmonic. Therefore, we define

$$\eta_\rho = \begin{cases} \phi(\vec{x},t), & \rho = 1, \\ 0, & \rho = 2,3, \end{cases}$$

reducing equation (3.1.25) to

$$\Pi(\vec{x},t) = \frac{\partial \mathscr{L}}{\partial \dot{\phi}}; \tag{3.1.28}$$

consequently, we write

$$\left|\vec{\Pi}_i^c \cdot \hat{y}'\right| = \frac{\partial \mathscr{L}}{\partial \dot{\phi}_i} \tag{3.1.29}$$

and

$$\left|\vec{\Pi}_i^{c'} \cdot \hat{y}'\right| = \frac{\partial \mathscr{L}}{\partial \dot{\phi}_i'}. \tag{3.1.30}$$

Equations (3.1.29) and (3.1.30) give us a direct way of calculating the right hand side of (3.1.23). This directly leads to the question of how we determine the Lagrangian density $\mathscr{L}$? This question is one of the main themes of this paper. It actually shows the generality of our solution to the problem of hydrodynamic mass transport and the flexibility it has with respect to any corrections suggested about the forces in phase II. For, should it be necessary to consider new or additional forces, we can add an interaction Lagrangian density $\mathscr{L}_{\mathrm{I}}$ that would correspond to some of the forces that were active in phase II. So our formalism treats the particles as an integrated part of the fluid and whatever happens to the fluid and streamlines is going to show its influence as active forces on the particles.

Let us return to equation (3.1.26) which was a result we obtained by considering irrotational flow. Equation (3.1.26) leads us to (3.1.29) and (3.1.30). However, the formalism in terms of irrotational flow is simple and too restrictive. So we seek a formalism of fluid dynamics that includes rotational flow. Therefore, we demand that $\operatorname{curl} \vec{v} = \vec{\nabla} \times \vec{v}$, the vorticity, does not vanish identically. One way to do that is to introduce an extra term to the gradient of ϕ found in (3.1.26) that makes $\operatorname{curl} \vec{v}$ nonvanishing. Thus, we postulate that there exists a vector potential field, $\vec{A}(\vec{x}, t)$, responsible for the irrotational flow such that $\vec{v} = \vec{v}(\vec{\nabla}\phi, \vec{A})$ is a function of $\vec{\nabla}\phi$ and $\vec{A}$, and $\vec{\nabla} \times \vec{v} \equiv \vec{f}(\vec{x}, t)$. But from incompressibility condition we have $\vec{\nabla} \cdot \vec{v} = 0$. Therefore, we want the following conditions

a) $\vec{v} = \vec{v}(\vec{\nabla}\phi, \vec{A})$

b) $\vec{\nabla} \times \vec{v} = \vec{f}(\vec{x}, t) \neq 0$

c) $\vec{\nabla} \cdot \vec{v} = 0$

to be met. If we try to write $\vec{v}(\vec{x}, t) = -\vec{\nabla}\phi(\vec{x}, t) + \vec{A}(\vec{x}, t)$, then (b) and (c) would imply $\vec{\nabla} \cdot \vec{A} = 0$ and $\vec{\nabla} \times \vec{A}(\vec{x}, t) = \vec{f}(\vec{x}, t)$. However, this system is overdetermined. We therefore look for a term that involves $\vec{A}(\vec{x}, t)$ and gives condition (c) automatically and condition (b) as the result of solving a differential equations problem. This immediately brings the term $\vec{\nabla} \times \vec{A}(\vec{x}, t)$ into our formalism. Therefore, we write

$$\vec{v}(\vec{x}, t) = -\vec{\nabla}\phi(\vec{x}, t) + \vec{\nabla} \times \vec{A}(\vec{x}, t). \tag{3.1.31}$$

Equation (3.1.31) is a very strong statement about the physics of hydrodynamic flow. It will, as we shall see, lead to a new theory and new interpretations of phenomena concerning hydrodynamic flow. One of the immediate consequences of (3.1.31) is that

$$\begin{aligned} \vec{\nabla} \times \vec{v} &= \vec{\nabla} \times [\vec{\nabla} \times \vec{A}(\vec{x}, t)] \\ &= \vec{\nabla}(\vec{\nabla} \cdot \vec{A}) - \nabla^2 \vec{A} \neq 0, \end{aligned}$$

that is, we are adding rotational motion (condition (b)), expressed as a nonvanishing curl of $\vec{v}$, to the dynamics of the flow. The incompressibility condition recovers equation (3.1.27) from (3.1.31), when $\vec{\nabla} \cdot (\vec{\nabla} \times \vec{A}) = 0$. Therefore, we do not have to worry about the scalar potential field ϕ having a more complicated form resulting from the curl of $\vec{A}$ added in (3.1.31), that is, ϕ still obeys the Laplace equation.

Equation (3.1.27) determines the form of the Lagrangian density $\mathscr{L}$ that is written in terms of ϕ. The next thing to do is to find an equation that governs the vector potential field $\vec{A}$ and then construct a Lagrangian density that corresponds to that equation. The final Lagrangian density would then be the sum of the one obtained from equation (3.1.27) and the one written in terms of $\vec{A}(\vec{x},t)$. Any other interaction Lagrangian density can then be added to this final Lagrangian density. Let us now try to find a vector potential field equation of motion that governs the postulated $\vec{A}(\vec{x},t)$.

From J. Serrin [25] the translation of the point $\vec{x}$ to the point $\vec{y}$ is described by the displacement functions $d_i(\vec{x},t)$, $i = 1,2,3$. Where $\vec{d}(\vec{x},t) = \vec{y} - \vec{x} = \sum_{i=1}^{3} d_i(\vec{x},t)\hat{e}_i$. Accordingly, we write $\vec{y} = \vec{d}(\vec{x},t)$ which leads to $\vec{x} = \vec{D}(\vec{y},t)$, where $\vec{D}$ is a function that translates $\vec{y}$ to $\vec{x}$. But

$$\frac{d\vec{y}}{dt} = \frac{\partial\vec{d}(\vec{x},t)}{\partial t} = \vec{v}(\vec{y},t) = \vec{v}(\vec{d}(\vec{x},t),t)$$

or

$$\frac{\partial\vec{d}(\vec{x},t)}{\partial t} = \vec{v}(\vec{x},t). \tag{3.1.32}$$

Equation (3.1.32) is significant because it expresses the fluid velocity in terms of the time derivative of the displacement of a fluid point. Now, we write the displacement vector as

$$\vec{d}(\vec{x},t) = \vec{d}_{||}(\vec{x},t) + \vec{d}_{\perp}(\vec{x},t), \tag{3.1.33}$$

where $\vec{d}_{||}(\vec{x},t)$ is the longitudinal displacement and $\vec{d}_{\perp}(\vec{x},t)$ is the transverse displacement. We substitute (3.1.33) into (3.1.32) to get

$$\vec{v}(\vec{x},t) = \frac{\partial}{\partial t}(\vec{d}_{||} + \vec{d}_{\perp}) = \frac{\partial\vec{d}_{||}}{\partial t} + \frac{\partial\vec{d}_{\perp}}{\partial t}. \tag{3.1.34}$$

Equating (3.1.34) and (3.1.31) yields

$$-\vec{\nabla}\phi(\vec{x},t) + \vec{\nabla} \times \vec{A}(\vec{x},t) = \frac{\partial \vec{d}_{\parallel}}{\partial t} + \frac{\partial \vec{d}_{\perp}}{\partial t}.$$

We now make the following identifications which will later be shown to be consistent with our mathematical development. These are

$$-\vec{\nabla}\phi = \frac{\partial \vec{d}_{\parallel}}{\partial t}, \tag{3.1.35}$$

and

$$\vec{\nabla} \times \vec{A} = \frac{\partial \vec{d}_{\perp}}{\partial t}. \tag{3.1.36}$$

From Fetter and Walecka [11] the Fourier transform of $\vec{d}(\vec{x},t)$ is

$$\vec{d}(\vec{x},t) = \frac{1}{(2\pi)^3} \int e^{i\vec{k}\cdot\vec{x}} \vec{\mathbb{D}}(\vec{k},t) d^3k \tag{3.1.37}$$

with the Fourier amplitude

$$\vec{\mathbb{D}}(\vec{k},t) = \int e^{-i\vec{k}\cdot\vec{x}} \vec{d}(\vec{x},t) d^3x.$$

For each $\vec{k}$ introduce a complete orthonormal set of unit vectors as shown in Figure (4).

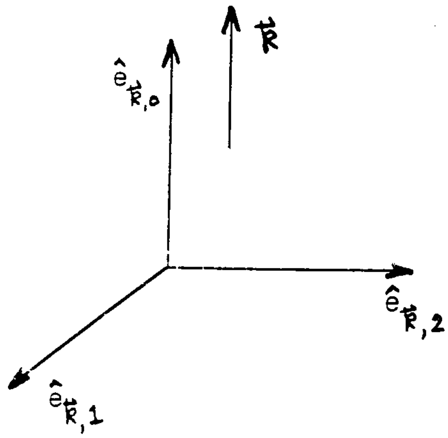

Figure 4: Complete orthonormal set of unit vectors for each Fourier component $\vec{k}$.

While $\hat{e}_{\vec{k},0} \equiv \vec{k}/|\vec{k}| = \hat{k}$ lies along the direction of $\vec{k}$, $\hat{e}_{\vec{k},1}$ and $\hat{e}_{\vec{k},2}$ are orthogonal to this direction. Expanding the Fourier amplitude in this complete basis set we get

$$\vec{\mathbb{D}}(\vec{k},t) = \sum_{s=0}^{2} \hat{e}_{\vec{k},s} a(\vec{k},s;t).$$

Substituting this expression into (3.1.37), we get

$$\vec{d}(\vec{x},t) = \vec{d}_{||}(\vec{x},t) + \vec{d}_{\perp}(\vec{x},t)$$

with

$$\vec{d}_{||}(\vec{x},t) = \frac{1}{(2\pi)^3} \int \hat{e}_{\vec{k},0} a(\vec{k},0;t) e^{i\vec{k}\cdot\vec{x}}\, d^3k \tag{3.1.38}$$

$$\vec{d}_{\perp}(\vec{x},t) = \frac{1}{(2\pi)^3} \int \sum_{s=1}^{2} \hat{e}_{\vec{k},s} a(\vec{k},s;t) e^{i\vec{k}\cdot\vec{x}}\, d^3k. \tag{3.1.39}$$

Direct differentiation of the above two expressions shows that

$$\vec{\nabla} \times \vec{d}_{||} = 0, \tag{3.1.40}$$

$$\vec{\nabla} \cdot \vec{d}_\perp = 0. \tag{3.1.41}$$

Equations (3.1.40) and (3.1.41) shows that the equalities we introduced in (3.1.35) and (3.1.36) are as we expected consistent. This is due to the fact that if we are to take the curl of (3.1.35) and the divergence of (3.1.36) then we will recover (3.1.40) and (3.1.41), that is

$$\vec{\nabla} \times (-\vec{\nabla}\phi) = \vec{\nabla} \times \frac{\partial \vec{d}_{\|}}{\partial t} = \frac{\partial}{\partial t}\vec{\nabla} \times \vec{d}_{\|}.$$

But $\vec{\nabla} \times \vec{\nabla}\phi = 0$, which implies that $\vec{\nabla} \times \vec{d}_{\|} = 0$. Similarly,

$$\vec{\nabla} \cdot (\vec{\nabla} \times \vec{A}) = \vec{\nabla} \cdot \frac{\partial \vec{d}_\perp}{\partial t} = \frac{\partial}{\partial t}\vec{\nabla} \cdot \vec{d}_\perp;$$

however, $\vec{\nabla} \cdot (\vec{\nabla} \times \vec{A}) = 0$ indicating that $\vec{\nabla} \cdot \vec{d}_\perp = 0$ as we expected. From equation (3.1.36) we can write

$$\vec{d}_\perp(\vec{x}, t) = \int dt\, [\vec{\nabla} \times \vec{A}(\vec{x}, t)]. \tag{3.1.42}$$

We consider equation (3.1.42) to be very significant in describing the time development of $\vec{d}_\perp(\vec{x}, t)$ in terms of $\vec{A}$ for which we are expecting to find an equation of motion. Now from (3.1.34), (3.1.35), and (3.1.36) we make the following definitions:

$$\vec{v}_{\|} \equiv \frac{\partial \vec{d}_{\|}}{\partial t} = -\vec{\nabla}\phi, \tag{3.1.43}$$

$$\vec{v}_\perp \equiv \frac{\partial \vec{d}_\perp}{\partial t} = \vec{\nabla} \times \vec{A}. \tag{3.1.44}$$

Also from (3.1.40) and (3.1.41) we have

$$\vec{\nabla} \times \vec{v}_{\|} = \frac{\partial}{\partial t}\left(\vec{\nabla} \times \vec{d}_{\|}\right) = 0, \tag{3.1.45}$$

and

$$\vec{\nabla} \cdot \vec{v}_\perp = \frac{\partial}{\partial t}\left(\vec{\nabla} \cdot \vec{d}_\perp\right) = 0. \tag{3.1.46}$$

Form the Navier-Stokes' equation. We have

$$\rho\frac{\partial \vec{v}}{\partial t} + \rho(\vec{v} \cdot \vec{\nabla})\vec{v} = \rho\vec{f} - \vec{\nabla}p + \eta\nabla^2\vec{v}, \tag{3.1.47}$$

where we set $\vec{\nabla} \cdot \vec{v} = 0$, and where $\rho\vec{f}$ is the applied force density. Now rewrite $\vec{v}$ and $\vec{f}$ as

$$\vec{v} = \vec{v}_{\|} + \vec{v}_\perp \tag{3.1.48}$$

and

$$\vec{f} = \vec{f}_{\|} + \vec{f}_\perp. \tag{3.1.49}$$

Substitute (3.1.48) and (3.1.49) into (3.1.47) to get

$$\rho\frac{\partial \vec{v}_{\|}}{\partial t} + \rho\frac{\partial \vec{v}_\perp}{\partial t} + \rho(\vec{v} \cdot \vec{\nabla})\vec{v} = \eta\nabla^2\vec{v}_{\|} + \eta\nabla^2\vec{v}_\perp - \vec{\nabla}p + \rho\vec{f}_{\|} + \rho\vec{f}_\perp,$$

or

$$\rho\frac{\partial \vec{v}_\perp}{\partial t} - \eta\nabla^2\vec{v}_\perp - \rho\vec{f}_\perp = \eta\nabla^2\vec{v}_{\|} - \rho\frac{\partial \vec{v}_{\|}}{\partial t} - \rho(\vec{v} \cdot \vec{\nabla})\vec{v} - \vec{\nabla}p + \rho\vec{f}_{\|}. \tag{3.1.50}$$

Now take the divergence of (3.1.50) to get

$$\vec{\nabla} \cdot \left(\rho\frac{\partial \vec{v}_\perp}{\partial t} - \eta\nabla^2\vec{v}_\perp - \rho\vec{f}_\perp\right) = 0 = \vec{\nabla} \cdot \left(\eta\nabla^2\vec{v}_{\|} - \rho\frac{\partial \vec{v}_{\|}}{\partial t} - \rho(\vec{v} \cdot \vec{\nabla})\vec{v} - \vec{\nabla}p + \rho\vec{f}_{\|}\right). \tag{3.1.51}$$

From the left hand side of (3.1.51) we have

$$\vec{\nabla} \cdot \left(\rho\frac{\partial \vec{v}_\perp}{\partial t} - \eta\nabla^2\vec{v}_\perp - \rho\vec{f}_\perp\right) = 0.$$

This tells us that we can either choose $\vec{B} \equiv \rho \partial \vec{v}_\perp / \partial t - \eta \nabla^2 \vec{v}_\perp - \rho \vec{f}_\perp$ to be zero or choose $\vec{B} = \vec{\nabla} \times \vec{\Upsilon}$, where $\vec{\Upsilon}$ is a vector. We make the first choice and get

$$\rho \frac{\partial \vec{v}_\perp}{\partial t} = \eta \nabla^2 \vec{v}_\perp + \rho \vec{f}_\perp. \tag{3.1.52}$$

This is what Wangsness [30] said to be the step Maxwell assumed in electromagnetism to get his expression for the displacement current. By substituting (3.1.44) into (3.1.52), we can write

$$\rho \frac{\partial^2 \vec{d}_\perp}{\partial t^2} = \eta \nabla^2 (\vec{\nabla} \times \vec{A}) + \rho \vec{f}_\perp. \tag{3.1.53}$$

Now we write $\vec{f}_\perp = \vec{\nabla} \times \vec{\mathscr{F}}_\perp$ with $\vec{\mathscr{F}}_\perp$ as a vector with $(\text{velocity})^2$ dimensions. Since we want to consider non-conservative forces in the rotational case we let

$$\vec{f}_\perp = C_0 \vec{\nabla} \times \left(\frac{\partial \vec{\mathscr{F}}_\perp}{\partial t} \right), \tag{3.1.54}$$

where C_0 is a constant of proportionality. In other words, we allow the force to be time dependent. Recalling (3.1.42) and using (3.1.54) equation (3.1.53) can be written as

$$\rho \frac{\partial^2}{\partial t^2} \int \left[\vec{\nabla} \times \vec{A}(\vec{x}, t') \right] dt' = \vec{\nabla} \times (\eta \nabla^2 \vec{A}) + C_0 \vec{\nabla} \times \frac{\partial (\rho \vec{\mathscr{F}}_\perp)}{\partial t}$$

or

$$\vec{\nabla} \times \left[\rho \frac{\partial^2}{\partial t^2} \int \vec{A}(\vec{x}, t') dt' - \eta \nabla^2 \vec{A} - C_0 \frac{\partial (\rho \vec{\mathscr{F}}_\perp)}{\partial t} \right] = 0.$$

By making the same choice we made to get (3.1.52) we obtain

$$\rho \frac{\partial^2}{\partial t^2} \int \vec{A}(\vec{x}, t') dt' - \eta \nabla^2 \vec{A} = C_0 \frac{\partial (\rho \vec{\mathscr{F}}_\perp)}{\partial t}. \tag{3.1.55}$$

Since we want the value of $\vec{A}(\vec{x},t')$ at time t at which the differential equation (3.1.55) holds we write

$$\vec{A}(\vec{x},t') \equiv \vec{A}(\vec{x},t')\delta(t-t').$$

Then we have (3.1.55) as

$$\rho\frac{\partial^2}{\partial t^2}\int \vec{A}(\vec{x},t')dt'\,\delta(t-t') - \eta\nabla^2\vec{A} = C_0\frac{\partial(\rho\vec{\mathscr{F}}_\perp)}{\partial t}.$$

But

$$\int dt'\,\delta(t-t')\vec{A}(\vec{x},t') = \vec{A}(\vec{x},t)$$

which leads to

$$\rho\frac{\partial^2\vec{A}}{\partial t^2} - \eta\nabla^2\vec{A} = C_0\frac{\partial\left(\rho\vec{\mathscr{F}}_\perp\right)}{\partial t}. \tag{3.1.56}$$

Equation (3.1.56) is a great accomplishment, it is actually more than what we expected. Now, (3.1.42) can be applied to fluid systems without any concern about the form that $\vec{A}$ might have. This is because equation (3.1.56) specifies the form of $\vec{A}$. If we set $\vec{\mathscr{F}}_\perp \equiv 0$ we then have

$$\rho\frac{\partial^2\vec{A}}{\partial t^2} - \eta\nabla^2\vec{A} = 0. \tag{3.1.57}$$

It turns out that (3.1.57) is as important as (3.1.56) in interpreting physical phenomena. (3.1.56) was the main equation we were after and now we can construct the Lagrangian that corresponds to (3.1.56) and add it to the Lagrangian that corresponds to (3.1.27). Let us now consider some of the physical consequences that we can deduce from equations (3.1.56), (3.1.57), and (3.1.31), which are results we obtain due to our consideration of rotational flow.

3.2 Consequences of rotational flow

We will start with equation (3.1.56). First note that $\vec{f}_\perp$ in (3.1.54) has the units of acceleration. As a result the quantity $\partial/\partial t\left(\rho\vec{\mathscr{F}}_\perp\right)$ has the units of mass density multiplied by velocity squared divided by time. In other words, the time rate of change of (mass density) $\times$ (velocity)^2. But $(d/dt)mv^2 = (d/dt)2E$. Where we used m for mass, v for velocity, and E for kinetic energy. We therefore identify the quantity $\partial(\rho\vec{\mathscr{F}}_\perp)/\partial t$ as the vector analog of the time rate of change of twice the kinetic energy density. However, we can also write $dE/dt = \vec{v}\cdot\vec{F}$. Then

$$\frac{\partial\left(\rho\vec{\mathscr{F}}_\perp\right)}{\partial t} = 2\vec{v}\cdot\vec{F} \tag{3.2.58}$$

with $\vec{F}$ as the force responsible for the dissipation of kinetic energy. Writing (3.1.56) in terms of (3.2.58) we have

$$\rho\frac{\partial^2\vec{A}}{\partial t^2} - \eta\nabla^2\vec{A} = C_0 2\vec{v}\cdot\vec{F}. \tag{3.2.59}$$

Since the left hand side of (3.2.59) is a vector quantity, the quantity $2\vec{v}\cdot\vec{F}$ should also be a vector quantity. Hence, we let $\vec{F}$ be a dyad or a tensor of second rank and rewrite it as $\mathbb{F}$. Then $2\vec{v}\cdot\mathbb{F}$ is a vector. We then have

$$\mathbb{F} = \begin{pmatrix} F_{11} & F_{12} & F_{13} \\ F_{21} & F_{22} & F_{23} \\ F_{31} & F_{32} & F_{33} \end{pmatrix} \tag{3.2.60}$$

which will make the term $2\vec{v} \cdot \mathbb{F}$ become

$$\begin{aligned} 2\vec{v} \cdot \mathbb{F} &= 2 \begin{pmatrix} + F_{11}\hat{e}_1\hat{e}_1 + F_{12}\hat{e}_1\hat{e}_2 + F_{13}\hat{e}_1\hat{e}_3 \\ + F_{21}\hat{e}_2\hat{e}_1 + F_{22}\hat{e}_2\hat{e}_2 + F_{23}\hat{e}_2\hat{e}_3 \\ + F_{31}\hat{e}_3\hat{e}_1 + F_{32}\hat{e}_3\hat{e}_2 + F_{33}\hat{e}_3\hat{e}_3 \end{pmatrix} \cdot (v_1\hat{e}_1 + v_2\hat{e}_2 + v_3\hat{e}_3) \\ &= (v_1F_{11} + v_2F_{12} + v_3F_{13})\hat{e}_1 + (v_1F_{21} + v_2F_{22} + v_3F_{23})\hat{e}_2 \\ &\quad + (v_1F_{31} + v_2F_{32} + v_3F_{33})\hat{e}_3 \end{aligned}$$

where we used the fact that $\vec{v} \cdot \mathbb{F} = \mathbb{F} \cdot \vec{v}$. Now we choose a certain form of the force tensor, $\mathbb{F}$, and substitute it into the above expression. This $\mathbb{F}$ would be one with the following components

$$F_{ij} = \eta \frac{\partial v_j}{\partial x_i} \qquad i, j = 1, 2, 3, \tag{3.2.61}$$

where η is the coefficient of viscosity. According to the force form in (3.2.61) the term $2\vec{v} \cdot \mathbb{F}$ is

$$\begin{aligned} 2\vec{v} \cdot \mathbb{F} &= \left[\eta \frac{\partial (v_1^2)}{\partial x_1} + \eta \frac{\partial (v_2^2)}{\partial x_1} + \eta \frac{\partial (v_3^2)}{\partial x_1}\right] \hat{e}_1 \\ &\quad + \left[\eta \frac{\partial (v_1^2)}{\partial x_2} + \eta \frac{\partial (v_2^2)}{\partial x_2} + \eta \frac{\partial (v_3^2)}{\partial x_2}\right] \hat{e}_2 \\ &\quad + \left[\eta \frac{\partial (v_1^2)}{\partial x_3} + \eta \frac{\partial (v_2^2)}{\partial x_3} + \eta \frac{\partial (v_3^2)}{\partial x_3}\right] \hat{e}_3, \end{aligned}$$

or

$$\begin{aligned} 2\vec{v} \cdot \mathbb{F} &= \eta \vec{\nabla} v_1^2 + \eta \vec{\nabla} v_2^2 + \eta \vec{\nabla} v_3^2 = \eta \vec{\nabla}(\vec{v} \cdot \vec{v}) \\ &= \vec{\nabla}(\eta v^2), \end{aligned}$$

that is,

$$2\vec{v}\cdot\mathbb{F} = \vec{\nabla}(\eta v^2) = \vec{\nabla}\left[\eta(\vec{v}\cdot\vec{v})\right]. \tag{3.2.62}$$

Equation (3.2.59) is then

$$\rho\frac{\partial^2\vec{A}}{\partial t^2} - \eta\nabla^2\vec{A} = C_0\vec{\nabla}(\eta v^2), \tag{3.2.63}$$

which adds to the of physical insights of equation (3.1.42). Equation (3.2.63) is written in components as

$$\rho\frac{\partial^2 A_i}{\partial t^2} - \eta\nabla^2 A_i = C_0\frac{\partial}{\partial x_i}\left(\sum_{j=1}^{3}\eta v_j^2\right) \qquad i = 1,2,3.$$

The right hand side form of (3.2.58) that we will work with in (3.1.23) is a purely mathematical one. However, it will illustrate the procedure and the ideas behind our approach. Later we will write the right hand side of (3.2.58) in the form

$$\rho\sum_{j=1}^{3}\dot{\alpha}_{ij}A_j + \lambda\sum_{j=1}^{3}\sum_{\ell=1}^{2}t_\ell\frac{\partial\dot{\alpha}_{ij}}{\partial x_\ell}A_j \qquad i = 1,2,3, \tag{3.2.64}$$

where the α_{ij} are friction related coefficients that are functions of position and time, and λ is a constant. We will come back to (3.2.64) later but at this point let us explore an important consequence of (3.1.57).

We will be able to explain the appearance of sand ripples on ocean beaches in terms of the vector potential field $\vec{A}$ that obeys equation (3.1.57). At this point we know that $\vec{A}$ is responsible for rotational flow. This $\vec{A}$ field in (3.1.57) has the solution

$$\vec{A}(\vec{x},t) = \vec{A}_0 e^{i(\vec{k}\cdot\vec{x}-\omega t)}, \tag{3.2.65}$$

where $\vec{A}_0$ is the amplitude, $\vec{k}$ propagation wave vector, and ω is the frequency of of oscillation. From (3.1.31) we have $\vec{\nabla} \times \vec{v} = \vec{\nabla} \times (\vec{\nabla} \times \vec{A})$, which means taking the curl of (3.2.65) twice, so we do that componentwise. Let $A_k = A_{0k} \exp i(\vec{k} \cdot \vec{x} - \omega t)$ be the kth component of $\vec{A}$. Taking the lth partial of the kth component would then yield, with $\partial/\partial x_\ell \equiv \partial_\ell$,

$$\begin{aligned} \partial_\ell A_k &= A_{0k}(ik_\ell)e^{i(\vec{k}\cdot\vec{x}-\omega t)} \\ &= (ik_\ell)A_k, \end{aligned}$$

which will lead to

$$\vec{\nabla} \times \vec{A} = \begin{vmatrix} \hat{e}_1 & \hat{e}_2 & \hat{e}_3 \\ \partial_1 & \partial_2 & \partial_3 \\ A_1 & A_2 & A_3 \end{vmatrix} = i\vec{k} \times \vec{A},$$

and

$$\begin{aligned} \vec{\nabla} \times (\vec{\nabla} \times \vec{A}) &= \vec{\nabla} \times (i\vec{k} \times \vec{A}) \\ &= (\vec{\nabla} \cdot \vec{A})(i\vec{k}) - (\vec{\nabla} \cdot i\vec{k})\vec{A} - (i\vec{k} \cdot \vec{\nabla})\vec{A} + (\vec{A} \cdot \vec{\nabla})i\vec{k}. \end{aligned}$$

Since $\vec{k}$ is a constant vector, we get

$$\vec{\nabla} \times (\vec{\nabla} \times \vec{A}) = \sum_{\ell=1}^{3} \frac{\partial A_\ell}{\partial x_\ell} i\vec{k} - \left(i \sum_{\ell=1}^{3} k_\ell \frac{\partial}{\partial x_\ell} \right) \vec{A}. \qquad (3.2.66)$$

But from the determinant above

$$\vec{\nabla} \times \vec{v} = \vec{\nabla} \times (\vec{\nabla} \times \vec{A}) = \vec{\nabla} \times (i\vec{k} \times \vec{A}).$$

Now let $\vec{v} = v_x(z)\hat{x}$. Then

$$\vec{\nabla} \times \vec{v} = (\vec{\nabla} \times \vec{v})_y = \hat{z}\frac{\partial v_x(z)}{\partial y} - \hat{y}\frac{\partial v_x(z)}{\partial z} = -\frac{\partial v_x(z)}{\partial z}\hat{y}. \tag{3.2.67}$$

From Fetter and Walecka [11], the drag force F_x acting on the fluid lying below the element of area da is $F_x = \eta(\partial v_x/\partial z)da$, with η as defined before. Then

$$\mathcal{F}_x = \eta(\partial v_x/\partial z) \tag{3.2.68}$$

is the force per unit area.

From (3.2.67) and (3.2.68), we see that $\mathcal{F}_x$ is the y component of $-\eta(\vec{\nabla} \times \vec{v})$ or, using the determinant expansion,

$$\mathcal{F}_x = -\eta\left[\vec{\nabla} \times (i\vec{k} \times \vec{A})\right] \cdot \hat{y}. \tag{3.2.69}$$

From (3.2.66) we obtain

$$\begin{aligned}
\mathcal{F}_x &= -\eta\left[\sum_{\ell=1}^{3}\frac{\partial A_\ell}{\partial x_\ell}ik_2 - \left(i\sum_{\ell=1}^{3}k_\ell\frac{\partial}{\partial x_\ell}\right)A_2\right] \\
&= -\eta\left[\sum_{\ell=1}^{3}(ik_\ell A_\ell)ik_2 - i\sum_{\ell=1}^{3}k_\ell ik_\ell A_2\right] \\
&= \left[\eta(\vec{k}\cdot\vec{A})k_y - (\vec{k}\cdot\vec{k})A_y\right].
\end{aligned}$$

But $\vec{k}\cdot\vec{A} = \vec{k}\cdot\vec{A}_0\exp[i(\vec{k}\cdot\vec{x} - \omega t)]$ and $A_y = A_{0y}\exp[i(\vec{k}\cdot\vec{x} - \omega t)]$, so

$$\begin{aligned}
\mathcal{F}_x &= \eta\left[(\vec{k}\cdot\vec{A}_0)k_y\exp[i(\vec{k}\cdot\vec{x} - \omega t)] - k^2A_{0y}\exp[i(\vec{k}\cdot\vec{x} - \omega t)]\right] \\
&= \eta\left[(\vec{k}\cdot\vec{A}_0)k_y - k^2A_{0y}\right]e^{i(\vec{k}\cdot\vec{x}-\omega t)},
\end{aligned}$$

or

$$\mathcal{F}_x = \eta_{eff} \exp[i(\vec{k} \cdot \vec{x} - \omega t)], \tag{3.2.70}$$

where $\eta_{eff} \equiv \eta \left[(\vec{k} \cdot \vec{A}_0)k_y - k^2 A_{0y}\right]$. Equation (3.2.70) says that the force per unit area on ocean beaches, or desert sands, is periodic in time and space, in other words, periodic solutions exist, which explains the phenomenon of wavy sand marks (ripples) on ocean beaches. We have therefore demonstrated two separate physical phenomena that come out as a consequence of (3.1.56) and (3.1.57). If we are to put the two equations together we would have to explain the second phenomenon (the ripples) using equation (3.1.56). Equation (3.1.56) is

$$\rho \frac{\partial^2 \vec{A}}{\partial t^2} - \eta \nabla^2 \vec{A} = C_0 \frac{\partial \left(\rho \vec{\mathscr{F}}_\perp\right)}{\partial t}.$$

From (3.2.63) we have

$$\rho \frac{\partial^2 \vec{A}}{\partial t^2} - \eta \nabla^2 \vec{A} = C_0 \vec{\nabla}(\eta v^2),$$

which has the solution $\vec{A} = \vec{A}_h + \vec{A}_p$, where $\vec{A}_h$ corresponds to the homogeneous part and $\vec{A}_p$ is the particular solution that gives a non zero right hand side in (3.2.63). The solution to (3.2.63) can be written as

$$\vec{A} = \vec{A}_0 \exp[i(\vec{k} \cdot \vec{x} - \omega t)] + \vec{A}_p \left(\vec{\nabla}(\eta v^2)\right),$$

which says that the $\vec{A}_p$ is responsible for the disappearance of ripples as we go further into the sea.

Finally, we take up equation (3.1.31). This equation defines a new vector field, $\vec{A}(\vec{x}, t)$, and makes the fluid flow rotational. It turns out that the oscillatory boundary condition of the fluid surface arises from the nonvanishing curl of the fluid velocity. The oscillatory boundary condition of a fluid surface is a phenomenon we observe daily; however, the mathematical boundary condition imposed on the differential equation governing the motion of fluids is the no slip condition, that is, v vanishes at the boundary. Therefore, we end up with a wavy surface with fixed end points, a situation more appropriate for strings rather than wavy fluid surfaces. Here, by virtue of the new vector field $\vec{A}$, we present a mathematical analysis that allows the replacement, as well as the recovery, of the no slip boundary condition and imposes an oscillatory fluid velocity as a boundary condition. This is the first nonphenomenological argument available, which allows one to relax the no slip condition. We observe no work done to replace the no slip boundary condition, imposed by Stokes', with a boundary condition that is physically appropriate. Since the boundary of a fluid moves in a wavy manner, physical appropriateness here means an oscillatory boundary condition. Serrin [25] mentioned in his article that Stokes argued that the fluid must adhere to the solid, since the contrary assumption implies an infinitely greater resistance to the sliding of one portion of the fluid past another than to the sliding of fluid over a solid. However, as we mentioned, the no slip condition is not always observed physically and that is sufficient proof that the no slip is not enough as a boundary condition. Let us now go through the mathematical analysis of the boundary situation. From R. Wangsness [30] we define the following unit vectors appropriate for the figure shown below, where $\hat{n}' = \hat{n} \times \hat{t}$, $\hat{t} = \hat{n}' \times \hat{n}$, $\hat{n} = \hat{t} \times \hat{n}'$.

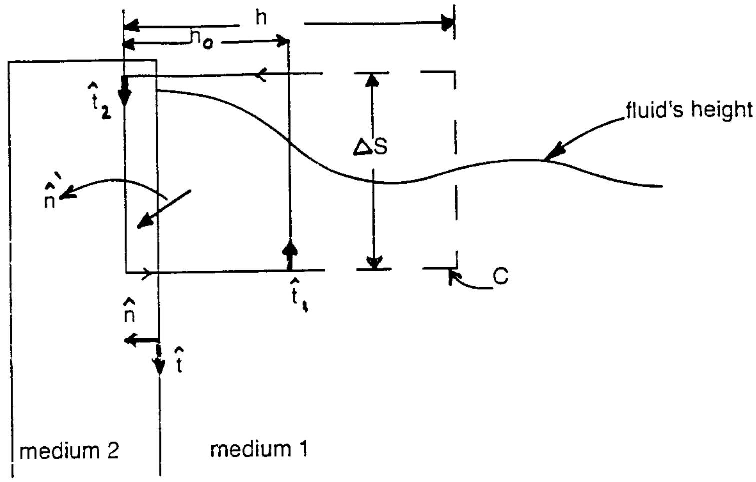

Figure 5: Showing the path C, the unit vectors, and the limit of h.

Now integrate $\vec{v}$ around the curve C in Figure (5) to get, by Stokes' theorem,

$$\begin{aligned}\oint_C \vec{v}\cdot d\vec{s} &= \vec{v}_2\cdot\hat{t}_2\Delta S + \vec{v}_1\cdot\hat{t}_1\Delta S + \mathscr{W}\\ &= \hat{t}\cdot(\vec{v}_2-\vec{v}_1)\Delta S + \mathscr{W}\\ &= \iint(\vec{\nabla}\times\vec{v})\cdot d\vec{a},\end{aligned}$$

where $\vec{v_1}$ is the velocity in medium 1 and $\vec{v_2}$ is the velocity in medium 2. Now $\mathscr{W}$ above is the contribution to the line integral from the sides of length h and ΔS is taken to be very small. Let $\Delta S = \epsilon/h$ with $|\epsilon| \ll 1$, then $h\Delta S = (\epsilon/h)h = \epsilon \ll 1$; therefore, the variation of the integrand over the area is small and we shall take it to be constant. Another thing to notice is that $\mathscr{W} = \int \vec{v}\cdot\hat{n}dS + \int \vec{v}'\cdot\hat{n}dS = \int(\vec{v}-\vec{v}')\cdot\hat{n}dS$, where $\vec{v}$ is the velocity at the top of the line segment ΔS and $\vec{v}'$ is the velocity at the bottom of ΔS. However, due to the smallness of the ΔS we considered, $\vec{v}\approx\vec{v}'$ which means that $\mathscr{W}\approx 0$ or nearly so. Our integral around the curve C becomes

$$\hat{t}\cdot(\vec{v}_2-\vec{v}_1)\Delta S = (\vec{\nabla}\times\vec{v})\cdot\hat{n}'h\Delta S. \tag{3.2.71}$$

To get the velocity at the boundary, we take the limit of (3.2.71) as h goes to h_0. This is due to the fact that the change in the velocity at the boundary is not instantaneous as it is in the case of an electrical property across a boundary, but rather gradual. Figure (5) shows how h_0 can be taken to be wave length dependent. From (3.2.71), we have

$$\lim_{h \to h_0} \hat{t} \cdot (\vec{v}_2 - \vec{v}_1) = \lim_{h \to h_0} (\vec{\nabla} \times \vec{v}) \cdot \hat{n}' h. \tag{3.2.72}$$

Now $\hat{t} = \hat{n}' \times \hat{n}$, so we can write (3.2.72) as

$$\hat{n}' \times \hat{n} \cdot (\vec{v}_2 - \vec{v}_1) = \lim_{h \to h_0} (\vec{\nabla} \times \vec{v}) \cdot \hat{n}' h$$

or

$$\begin{aligned} \left[\hat{n}' \cdot \hat{n} \times (\vec{v}_2 - \vec{v}_1) - \lim_{h \to h_0} (\vec{\nabla} \times \vec{v}) \cdot \hat{n}' h\right] &= 0, \\ \hat{n}' \cdot \left[\hat{n} \times (\vec{v}_2 - \vec{v}_1) - \lim_{h \to h_0} (\vec{\nabla} \times \vec{v}) h\right] &= 0 \end{aligned}$$

holds for the arbitrary direction of $\hat{n}'$, which implies

$$\hat{n} \times (\vec{v}_2 - \vec{v}_1) = \lim_{h \to h_0} h(\vec{\nabla} \times \vec{v}). \tag{3.2.73}$$

The velocity $\vec{v}$ can be written as

$$\vec{v} = \vec{v}_n + \vec{v}_\perp = v_n \hat{n} + v_\perp \hat{t}.$$

Consequently, we can write

$$\hat{n} \times \vec{v} = \hat{n} \times (v_n \hat{n} + \vec{v}_\perp) = v_n \hat{n} \times \hat{n} + \hat{n} \times \vec{v}_\perp = \hat{n} \times \vec{v}_\perp,$$

which implies that $\hat{n} \times \vec{v} = \hat{n} \times \vec{v}_{\perp}$ and that $\hat{n} \times (\vec{v}_2 - \vec{v}_1) = \hat{n} \times (\vec{v}_{2\perp} - \vec{v}_{1\perp})$. Therefore, (3.2.73) is

$$\hat{n} \times (\vec{v}_{2\perp} - \vec{v}_{1\perp}) = \lim_{h \to h_0} h(\vec{\nabla} \times \vec{v}); \tag{3.2.74}$$

but $(\hat{n} \times \vec{v}) \times \hat{n} = -\hat{n} \times (\hat{n} \times \vec{v}) = -\hat{n}(\hat{n} \cdot \vec{v}) + \vec{v}(\hat{n} \cdot \hat{n}) = \vec{v} - v_n \hat{n} = v_n \hat{n} + \vec{v}_{\perp} - v_n \hat{n} = \vec{v}_{\perp}$. So if we cross (3.2.74) from the right by $\hat{n}$ we get

$$\begin{aligned} [\hat{n} \times (\vec{v}_{2\perp} - \vec{v}_{1\perp})] \times \hat{n} &= \lim_{h \to h_0} h(\vec{\nabla} \times \vec{v}) \times \hat{n} \\ (\hat{n} \times \vec{v}_{2\perp}) \times \hat{n} - (\hat{n} \times \vec{v}_{1\perp}) \times \hat{n} &= \lim_{h \to h_0} h(\vec{\nabla} \times \vec{v}) \times \hat{n} \end{aligned}$$

which allows us to rewrite (3.2.74) as

$$\vec{v}_{2\perp} - \vec{v}_{1\perp} = \lim_{h \to h_0} [h(\vec{\nabla} \times \vec{v}) \times \hat{n}]. \tag{3.2.75}$$

According to the path C in Figure (5) we have $\vec{v}_{2\perp} = 0$ and as a result of that, we rewrite the vector $v_{1\perp}$ as $v_{\perp}$. So (3.2.75) becomes

$$\vec{v}_{\perp} = \lim_{h \to h_0} \{h[\hat{n} \times (\vec{\nabla} \times \vec{v})]\}. \tag{3.2.76}$$

But $\vec{\nabla} \times \vec{v} = \vec{\nabla} \times (-\vec{\nabla}\phi + \vec{\nabla} \times \vec{A}) = \vec{\nabla} \times (\vec{\nabla} \times \vec{A})$. From equation (3.1.36) we have $\vec{\nabla} \times \vec{A} = \partial \vec{d}_{\perp}/\partial t$ so that $\vec{\nabla} \times \vec{v} = \vec{\nabla} \times \left(\partial \vec{d}_{\perp}/\partial t\right)$ and from equation (3.1.37), we have

$$\vec{\nabla} \times \vec{v} = \vec{\nabla} \times \frac{1}{(2\pi)^3} \int e^{i\vec{k}\cdot\vec{x}} \frac{\partial \vec{\mathbb{D}}(\vec{k}, t)}{\partial t} d^3k.$$

Take $\vec{k} = -k_y \hat{y} = k_0 \hat{e}_{\vec{k},0} + 0 \cdot \hat{e}_{\vec{k},1} + 0 \cdot \hat{e}_{\vec{k},2}$, that is, $\hat{e}_{\vec{k},0} = -\hat{y}$, $\hat{e}_{\vec{k},1} = \hat{x}$, $\hat{e}_{\vec{k},2} = \hat{z}$. The incoming wave will then have a unit vector $\hat{k}$ in the direction opposite to $\hat{y}$.

Now we know from (3.1.39) that

$$\begin{aligned}\vec{d}_\perp(\vec{x},t) &= \frac{1}{(2\pi)^3}\int\sum_{s=1}^{2}\hat{e}_{\vec{k},s}a(\vec{k},s;t)e^{i\vec{k}\cdot\vec{x}}d^3k \\ &= \frac{1}{(2\pi)^3}\int d^3k\left[\hat{e}_{\vec{k},1}a(\vec{k},1;t)+\hat{e}_{\vec{k},2}a(\vec{k},2;t)\right]e^{i\vec{k}\cdot\vec{x}}.\end{aligned}$$

This will help us calculate $\vec{\nabla}\times(\partial\vec{d}_\perp/\partial t)$ which we will fit into (3.2.76); therefore,

$$\begin{aligned}\vec{\nabla}\times\left(\frac{\partial\vec{d}_\perp}{\partial t}\right) &= \left(\hat{x}\frac{\partial}{\partial x}+\hat{y}\frac{\partial}{\partial y}+\hat{z}\frac{\partial}{\partial z}\right)\times\frac{1}{(2\pi)^3}\int d^3k\,[\hat{x}\dot{a}_1+\hat{z}\dot{a}_2]\,e^{i\vec{k}\cdot\vec{x}} \\ &= \frac{1}{(2\pi)^3}\int d^3k\left[\hat{x}\frac{\partial}{\partial y}\dot{a}_2e^{i\vec{k}\cdot\vec{x}}-\hat{z}\frac{\partial}{\partial y}\dot{a}_1e^{i\vec{k}\cdot\vec{x}}\right. \\ &\quad\left.-\,\hat{y}\frac{\partial}{\partial x}\dot{a}_2e^{i\vec{k}\cdot\vec{x}}+\hat{y}\frac{\partial}{\partial z}\dot{a}_1e^{i\vec{k}\cdot\vec{x}}\right]\end{aligned}$$

with $\dot{a}_i\equiv\partial a_i/\partial t$, $i=1,2$. If $\vec{k}=-k_y\hat{y}$ and $k_x=k_z=0$, then

$$\vec{\nabla}\times\left(\frac{\partial\vec{d}_\perp}{\partial t}\right)=\frac{1}{(2\pi)^3}\int d^3k\,[-\hat{z}\dot{a}_1(ik_y)+\hat{x}\dot{a}_2(ik_y)]\,e^{i\vec{k}\cdot\vec{x}}. \tag{3.2.77}$$

By noting that $\hat{n}=-\hat{y}$, we can write (3.2.76) as

$$\vec{v}_\perp=\lim_{h\to h_0}h\left\{-\hat{y}\times\left[\vec{\nabla}\times\left(\frac{\partial\vec{d}_\perp}{\partial t}\right)\right]\right\}. \tag{3.2.78}$$

Now substitute (3.2.77) into (3.2.78) with $k_y=2\pi/\lambda_y$. Then we have

$$\begin{aligned}\vec{v}_\perp &= \lim_{h\to h_0}\left\{-\hat{y}\times\frac{2\pi i}{(2\pi)^3}\int d^3k\left[-\hat{z}\dot{a}_1\frac{h}{\lambda_y}+\hat{x}\dot{a}_2\frac{h}{\lambda_y}\right]e^{i\vec{k}\cdot\vec{x}}\right\} \\ &= -\hat{y}\times\left\{\frac{2\pi i}{(2\pi)^3}\int d^3k\left[-\hat{z}\dot{a}_1\frac{h_0}{\lambda_y}+\hat{x}\dot{a}_2\frac{h_0}{\lambda_y}\right]e^{i\vec{k}\cdot\vec{x}}\right\}.\end{aligned}$$

For $h_0 = \lambda_y$ we get

$$\vec{v}_\perp = -\hat{y} \times \left[\frac{2\pi i}{(2\pi)^3} \int d^3k(-\hat{z}\dot{a}_1 + \hat{x}\dot{a}_2)e^{i\vec{k}\cdot\vec{x}}\right]. \tag{3.2.79}$$

By taking the cross product we get

$$\begin{aligned}
\vec{v}_\perp &= 2\pi i \frac{1}{(2\pi)^3} \int d^3k(\hat{x}\dot{a}_1 + \hat{z}\dot{a}_2)e^{i\vec{k}\cdot\vec{x}} \\
&= 2\pi i \left[\frac{1}{(2\pi)^3} \int d^3k(\hat{x}\dot{a}_1 + \hat{z}\dot{a}_2)e^{i\vec{k}\cdot\vec{x}}\right] \\
&= 2\pi i \left(\frac{\partial \vec{d}_\perp}{\partial t}\right) = 2\pi i(\vec{\nabla} \times \vec{A}) \neq 0.
\end{aligned}$$

In other words, we removed the no slip boundary condition and imposed the above one for $\vec{v}_\perp$ by choosing $h_0 = \lambda_y$. The no slip condition is recovered for $h_0 \neq \lambda_y$. At this point it is obvious that we do not have to refer to the fluid's adherence to the solid wall, as Stokes argued, to have a no slip condition. The no slip condition is a rather mathematical consequence we obtain as we introduce the above inequality in the slip equation. Let us demonstrate the exact form of (3.2.79). Since

$$\vec{v}_\perp = 2\pi i(\vec{\nabla} \times \vec{A}), \tag{3.2.80}$$

then from (3.2.65) $\vec{A} = \vec{A}_0 e^{i(\vec{k}\cdot\vec{x}-\omega t)}$ with $\vec{k} \cdot \vec{x} = k_y \hat{y}$, we get

$$\begin{aligned}
\vec{\nabla} \times \vec{A} &= (\hat{x}\partial_x + \hat{y}\partial_y + \hat{z}\partial_z) \times (A_0\hat{x} + A_0\hat{y} + A_0\hat{z})e^{-ik_y y}e^{-i\omega t} \\
&= \left(-\hat{z}A_0\partial_y e^{-ik_y y} + \hat{x}A_0\partial_y e^{-ik_y y}\right) e^{-i\omega t} \\
&= \left[\hat{z}A_0(ik_y) + \hat{x}A_0(-ik_y)\right] e^{i(\vec{k}\cdot\vec{x}-\omega t)} \\
&= ik_y(-\hat{x}A_0 + \hat{z}A_0)e^{i(\vec{k}\cdot\vec{x}-\omega t)},
\end{aligned}$$

which, after substitution into (3.2.80), gives

$$\vec{v}_\perp = 2\pi k_y(\hat{x} - \hat{z})A_0 e^{i(\vec{k}\cdot\vec{x}-\omega t)}.$$

From R. Guenther and John A. Crow [14], we have $\partial h/\partial t = \vec{v}\cdot\hat{n}_f$, where h is the surface height, $\vec{v} = (v_1, v_2, v_3)$ is the velocity of the fluid, and $\hat{n}_f$ is the normal to the surface. In our case we have $\partial h/\partial t = \vec{v}_\perp \cdot \hat{n}_f$. Setting $\hat{n}_f = \hat{z}$, we get

$$\frac{\partial h}{\partial t} = \frac{-(2\pi)^2}{\lambda_y} A_0 e^{i(\vec{k}\cdot\vec{x}-\omega t)}. \tag{3.2.81}$$

Equation (3.2.81) is an oscillatory boundary condition demonstrating yet another success of (3.1.57) and (3.1.31). Note the inverse proportionality between height's velocity and wavelength of incoming water waves, which is a physically expected result.

3.3 Hydrodynamic forces with potential fields

Now that we have demonstrated some of the consequences of (3.1.31), (3.1.56), and (3.1.57), let us go back to the original problem, which was constructing a Lagrangian density that corresponds to (3.1.56) with (3.2.64) on its right hand side. Then we shall add this Lagrangian density to the one that corresponds to (3.1.27) to get a final Lagrangian density from which we can get the canonical momentum density. We then substitute this canonical momentum density into (3.1.23) to get an expression for (3.1.24) which is on the right hand side of (3.1.17) and (3.1.18) which represents our final solutions to the problem of finding the equations of motion that correspond to the coordinates of discrete particles subjected to hydrodynamic forces. The Lagrangian density that corresponds to (3.1.27) was found by G. B. Whitham [31] in his paper on variational methods and applications to water waves, is given by

$$\mathscr{L}_1 = \left(\dot{\phi} + \frac{1}{2}(\vec{\nabla}\phi)^2 + gz\right), \tag{3.3.82}$$

where g is the gravitational acceleration, and $\mathscr{L}_1$ is the Lagrangian density of the scalar field ϕ. As far as $\vec{A}(\vec{x},t)$ is concerned, its vector nature gives us three nonzero components of the canonical momentum density found from (3.1.25). From the previous notation above equation (3.1.28), we define

$$\eta_\rho = \begin{cases} A_1(\vec{x},t), & \text{if } \rho = 1, \\ A_2(\vec{x},t), & \text{if } \rho = 2, \\ A_3(\vec{x},t), & \text{if } \rho = 3, \end{cases}$$

and consequently

$$\Pi_\rho(\vec{x},t) = \frac{\partial \mathscr{L}_2}{\partial \dot{A}_\rho}, \qquad \rho = 1,2,3 \tag{3.3.83}$$

are the components of the canonical momentum density due to the presence of $\vec{A}(\vec{x},t)$. Here $\mathscr{L}_2$, in contrast to $\mathscr{L}_1$ in equation (3.3.82), refers to the vector potential field Lagrangian density, the variation of which yields the equation of motion of the vector potential field which by now we write as

$$\rho\frac{\partial^2 A_j}{\partial t^2} - \eta\nabla^2 A_j = \rho\sum_{n=1}^{3}\dot{\alpha}_{jn}A_n + \lambda\sum_{n=1}^{3}\sum_{\ell=1}^{2} t_\ell\frac{\partial\dot{\alpha}_{jn}}{\partial x_\ell}A_n, \tag{3.3.84}$$

with $j = 1,2,3$, where as stated earlier, we used (3.2.64) for the right hand side of (3.2.58). The final Lagrangian density is

$$\mathscr{L}_0 = \mathscr{L}_1 + \mathscr{L}_2. \tag{3.3.85}$$

In order for us to have a final canonical momentum density, we define the following four termed vector $\vec{\Pi}_k^c$ for the kth particle and call it the canonical momentum density

$$\vec{\Pi}_k^c = \sum_{j=0}^{3} \frac{\partial \mathscr{L}_0}{\partial \dot{\eta}_{jk}} \hat{e}_j, \tag{3.3.86}$$

where $\hat{e}_0 = \sum_{j=1}^{3} a_j \hat{e}_j$ with a_j being the direction cosines, and where for the kth particle

$$\eta_{jk} = \begin{cases} \phi_k, & j = 0, \\ A_{1k}, & j = 1, \\ A_{2k}, & j = 2, \\ A_{3k}, & j = 3. \end{cases}$$

Expanding (3.3.86) we get

$$\begin{aligned} \vec{\Pi}_k^c &= \sum_{j=0}^{3} \frac{\partial \mathscr{L}_0}{\partial \dot{\eta}_{jk}} \hat{e}_j = \frac{\partial \mathscr{L}_0}{\partial \dot{\eta}_{0k}} \hat{e}_0 + \frac{\partial \mathscr{L}_0}{\partial \dot{\eta}_{1k}} \hat{e}_1 + \frac{\partial \mathscr{L}_0}{\partial \dot{\eta}_{2k}} \hat{e}_2 + \frac{\partial \mathscr{L}_0}{\partial \dot{\eta}_{3k}} \hat{e}_3 \\ &= \Pi_0^c \hat{e}_0 + \Pi_1^c \hat{e}_1 + \Pi_2^c \hat{e}_2 + \Pi_3^c \hat{e}_3. \end{aligned}$$

The unit vector $\hat{e}_0$ is chosen to be in the direction of propagation of the canonical momentum density of the scalar potential field, $\phi_k(\vec{x}, t)$. Equations (3.1.29) and (3.1.30), respectively, become

$$\begin{aligned} \left|\vec{\Pi}_k^c \cdot \hat{y}'\right| &= \left|\sum_{j=0}^{3} \frac{\partial \mathscr{L}_0}{\partial \dot{\eta}_{jk}} \hat{e}_j \cdot \hat{y}'\right| = \left|\frac{\partial \mathscr{L}_0}{\partial \dot{\phi}_k} \hat{e}_0 \cdot \hat{y}' + \sum_{j=1}^{3} \frac{\partial \mathscr{L}_0}{\partial \dot{A}_{jk}} \hat{e}_j \cdot \hat{y}'\right| \\ &= \left|\sum_{j=1}^{3} \left(\frac{\partial \mathscr{L}_0}{\partial \dot{\phi}_k} a_j + \frac{\partial \mathscr{L}_0}{\partial \dot{A}_{jk}}\right) (\hat{e}_j \cdot \hat{y}')\right| \end{aligned} \tag{3.3.87}$$

and

$$\left|\vec{\Pi}_k^{c'} \cdot \hat{y}'\right| = \left|\sum_{j=0}^{3} \frac{\partial \mathscr{L}_0}{\partial \dot{\eta}'_{jk}} \hat{e}_j \cdot \hat{y}'\right| = \left|\frac{\partial \mathscr{L}_0}{\partial \dot{\phi}'_k} \hat{e}_0 \cdot \hat{y}' + \sum_{j=1}^{3} \frac{\partial \mathscr{L}_0}{\partial \dot{A}'_{jk}} \hat{e}_j \cdot \hat{y}'\right|$$

$$= \left| \sum_{j=1}^{3} \left(\frac{\partial \mathscr{L}_0}{\partial \dot{\phi}'_k} a'_j + \frac{\partial \mathscr{L}_0}{\partial \dot{A}'_{jk}} \right) (\hat{e}_j \cdot \hat{y}') \right| . \tag{3.3.88}$$

By substituting (3.3.87) and (3.3.88) into (3.1.23) and then rewriting (3.1.24), we will obtain the final form of our equations of motion, (3.1.17) and (3.1.18). But before we do that, we must specify the mathematical form of the Lagrangian density $\mathscr{L}_0$ appearing in (3.3.87) and (3.3.88). Equation (3.3.85) expresses this Lagrangian density as the sum of $\mathscr{L}_1$ and $\mathscr{L}_2$. Since $\mathscr{L}_1$ is given in (3.3.82), we must specify the mathematical form of $\mathscr{L}_2$. After properly specifying $\mathscr{L}_2$, the variation of the integral

$$I = \int dt \int d\vec{x} \mathscr{L}_2 \tag{3.3.89}$$

will yield equation (3.3.84). We propose the following form for $\mathscr{L}_2$ that corresponds to the jth component of $\vec{A}$

$$\mathscr{L}_2 = \left[\frac{1}{2}\rho(\dot{A}_j)^2 - \frac{1}{2}\eta(\vec{\nabla}A_j)^2 - \rho\sum_{n=1}^{3}\alpha_{jn}\dot{A}_n A_j - \lambda\sum_{n=1}^{3}(\hat{t}\cdot\vec{\nabla}\alpha_{jn})\dot{A}_n A_j \right], \tag{3.3.90}$$

where $\hat{t} = (t_1, t_2)$ is a unit vector in the xy-plane and

$$\hat{t}\cdot\vec{\nabla}\alpha_{jn}(\vec{x},t) = \sum_{\ell=1}^{2} t_\ell \frac{\partial \alpha_{jn}}{\partial x_\ell} = t_1\frac{\partial \alpha_{jn}}{\partial x_1} + t_2\frac{\partial \alpha_{jn}}{\partial x_2}. \tag{3.3.91}$$

where the α_{jn} are the components of the matrix arising from friction. To check the correctness of our Lagrangian density in (3.3.90) we must apply the variational technique and see whether we get (3.3.84) or not. Let us do that next.

Let us apply the variation to the first component in $\vec{A}$. Then we have the following transformations applied to the field A_1

$$A_1(\vec{x},t) \;\rightarrow\; \epsilon\zeta_1(\vec{x},t) + A_1(\vec{x},t)$$

$$\dot{A}_1(\vec{x},t) \quad \rightarrow \quad \epsilon\dot{\zeta}_1(\vec{x},t) + \dot{A}_1(\vec{x},t),$$

which will make the integral in (3.3.89) an ϵ dependent integral. Then we set

$$\delta I = \frac{d}{d\epsilon} I(\epsilon)\Big|_{\epsilon=0} = \int dt \int d\vec{x}\,\delta\mathscr{L}_2 = 0,$$

where ζ is equal to zero when $t = t_1$ or $t = t_2$. By substituting the transformed A_1 and $\dot{A}_1$ in $\mathscr{L}_2$, we can write

$$\begin{aligned}
\delta I \;=\; & \int dt \int d\vec{x}\,\delta\Big\{\frac{1}{2}\rho(\dot{A}_1 + \epsilon\dot{\zeta}_1)^2 - \frac{1}{2}\eta\left[\vec{\nabla}(A_1 + \epsilon\zeta_1)\right]^2 \\
& - \rho\alpha_{11}(\dot{A}_1 + \epsilon\dot{\zeta}_1)(A_1 + \epsilon\zeta_1) - \rho\alpha_{12}\dot{A}_2(A_1 + \epsilon\zeta_1) \\
& - \rho\alpha_{13}\dot{A}_3(A_1 + \epsilon\zeta_1) \\
& - \lambda\sum_{\ell=1}^{2} t_\ell\Big[\frac{\partial\alpha_{11}}{\partial x_\ell}(\dot{A}_1 + \epsilon\dot{\zeta}_1)(A_1 + \epsilon\zeta_1) + \frac{\partial\alpha_{12}}{\partial x_\ell}\dot{A}_2(A_1 + \epsilon\zeta_1) \\
& + \frac{\partial\alpha_{13}}{\partial x_\ell}\dot{A}_3(A_1 + \epsilon\zeta_1)\Big]\Big\}\Big|_{\epsilon=0} \\
=\; & 0.
\end{aligned} \tag{3.3.92}$$

The integrand in (3.3.92) becomes

$$\begin{aligned}
\left(\frac{\partial\mathscr{L}_2}{\partial\epsilon}\right)\Big|_{\epsilon=0} \;=\; & \Big[\rho(\dot{A}_1 + \epsilon\dot{\zeta}_1)\dot{\zeta}_1 - \eta\vec{\nabla}(A_1 + \epsilon\zeta_1)\cdot\vec{\nabla}\zeta_1 \\
& - \rho\alpha_{11}(\dot{A}_1 + \epsilon\dot{\zeta}_1)\zeta_1 - \rho\alpha_{11}\dot{\zeta}_1(A_1 + \epsilon\zeta_1) - \rho\alpha_{12}\dot{A}_2\zeta_1 \\
& - \rho\alpha_{13}\dot{A}_3\zeta_1 - \lambda\sum_{\ell=1}^{2} t_\ell\frac{\partial\alpha_{11}}{\partial x_\ell}(\dot{A}_1 + \epsilon\dot{\zeta}_1)\zeta_1 \\
& - \lambda\sum_{\ell=1}^{2} t_\ell\frac{\partial\alpha_{11}}{\partial x_\ell}\dot{\zeta}_1(A_1 + \epsilon\zeta_1) - \lambda\sum_{\ell=1}^{2} t_\ell\frac{\partial\alpha_{12}}{\partial x_\ell}\dot{A}_2\zeta_1 \\
& - \lambda\sum_{\ell=1}^{2} t_\ell\frac{\partial\alpha_{13}}{\partial x_\ell}\dot{A}_3\zeta_1\Big]\Big|_{\epsilon=0},
\end{aligned}$$

which leads to

$$
\begin{aligned}
\delta I &= \int dt \int d\vec{x} \Big[\rho \dot{A}_1 \dot{\zeta}_1 - \eta \left(\vec{\nabla} A_1 \cdot \vec{\nabla} \zeta_1 \right) - \rho \alpha_{11} \dot{A}_1 \zeta_1 - \rho \alpha_{11} \dot{\zeta}_1 A_1 \\
&\quad - \rho \alpha_{12} \dot{A}_2 \zeta_1 - \rho \alpha_{13} \dot{A}_3 \zeta_1 - \lambda \sum_\ell t_\ell \frac{\partial \alpha_{11}}{\partial x_\ell} \dot{A}_1 \zeta_1 \\
&\quad - \lambda \sum_\ell t_\ell \frac{\partial \alpha_{11}}{\partial x_\ell} \dot{\zeta}_1 A_1 - \lambda \sum_\ell t_\ell \frac{\partial \alpha_{12}}{\partial x_\ell} \dot{A}_2 \zeta_1 \\
&\quad - \lambda \sum_\ell t_\ell \frac{\partial \alpha_{13}}{\partial x_\ell} \dot{A}_3 \zeta_1 \Big] = 0. \qquad (3.3.93)
\end{aligned}
$$

Now integrate by parts and note that

$$
\int d\vec{x} \int dt \, \rho \alpha_{11} \dot{\zeta}_1 A_1 = - \int d\vec{x} \int dt \, \rho (\dot{\alpha}_{11} A_1 + \alpha_{11} \dot{A}_1) \zeta_1,
$$

so equation (3.3.93) becomes

$$
\begin{aligned}
\delta I &= \int dt \int d\vec{x} \Big[-\rho \ddot{A}_1 \zeta_1 + \eta \nabla^2 A_1 \zeta_1 + (\rho \dot{\alpha}_{11} A_1 + \rho \alpha_{11} \dot{A}_1) \zeta_1 \\
&\quad - \rho \alpha_{11} \dot{A}_1 \zeta_1 - \rho \alpha_{12} \dot{A}_2 \zeta_1 - \rho \alpha_{13} \dot{A}_3 \zeta_1 \\
&\quad - \lambda \sum_\ell t_\ell \frac{\partial \alpha_{11}}{\partial x_\ell} \dot{A}_1 \zeta_1 + \lambda \sum_\ell t_\ell \frac{\partial \dot{\alpha}_{11}}{\partial x_\ell} A_1 \zeta_1 + \lambda \sum_\ell t_\ell \frac{\partial \alpha_{11}}{\partial x_\ell} \dot{A}_1 \zeta_1 \\
&\quad - \lambda \sum_\ell t_\ell \frac{\partial \alpha_{12}}{\partial x_\ell} \dot{A}_2 \zeta_1 - \lambda \sum_\ell t_\ell \frac{\partial \alpha_{13}}{\partial x_\ell} \dot{A}_3 \zeta_1 \Big] = 0. \qquad (3.3.94)
\end{aligned}
$$

Cancelling the appropriate terms in (3.3.94) and factoring out the independent coordinate ζ_1, we get

$$
\begin{aligned}
\delta I &= \int dt \int d\vec{x} \Big[-\rho \ddot{A}_1 + \eta \nabla^2 A_1 + \rho \dot{\alpha}_{11} A_1 + \rho \dot{\alpha}_{12} A_2 + \rho \dot{\alpha}_{13} A_3 + \lambda \sum_\ell t_\ell \frac{\partial}{\partial x_\ell} \frac{\partial \alpha_{11}}{\partial t} A_1 \\
&\quad + \lambda \sum_\ell t_\ell \frac{\partial}{\partial x_\ell} \frac{\partial \alpha_{12}}{\partial t} A_2 + \lambda \sum_\ell t_\ell \frac{\partial}{\partial x_\ell} \frac{\partial \alpha_{13}}{\partial t} A_3 \Big] \zeta_1 = 0. \qquad (3.3.95)
\end{aligned}
$$

By the fundamental lemma of the calculus of variations, we set the coefficients equal to zero to obtain

$$\rho\ddot{A}_1 - \eta\nabla^2 A_1 = \rho\sum_{n=1}^{3}\dot{\alpha}_{1n}A_n + \lambda\sum_{n=1}^{3}\sum_{\ell=1}^{2}t_\ell\frac{\partial\dot{\alpha}_{1n}}{\partial x_\ell}A_n. \tag{3.3.96}$$

Taking the variations in A_2 and A_3 yield the corresponding equation of motion (3.3.96). Therefore, we have

$$\rho\frac{\partial^2 A_j}{\partial t^2} - \eta\nabla^2 A_j = \rho\sum_{n=1}^{3}\dot{\alpha}_{jn}A_n + \lambda\sum_{n=1}^{3}\sum_{\ell=1}^{2}t_\ell\frac{\partial\dot{\alpha}_{jn}}{\partial x_\ell}A_n, \quad j = 1,2,3, \tag{3.3.97}$$

which is precisely (3.3.84). This tells us that (3.3.90) is the correct Lagrangian density for our equation of motion (3.3.84). Equation (3.3.85), the final Lagrangian density for the kth particle, is then

$$\begin{aligned}\mathscr{L}_0 \;=\; & \Big[\dot{\phi}_k + \frac{1}{2}(\vec{\nabla}\phi_k)^2 + gz + \frac{1}{2}\rho(\dot{A}_{jk})^2 - \frac{1}{2}\eta(\vec{\nabla}A_{jk})^2 \\ & - \rho\sum_{n=1}^{3}\alpha_{jn}\dot{A}_{nk}A_{jk} - \lambda\sum_{n=1}^{3}(\hat{t}\cdot\vec{\nabla}\alpha_{jn})\dot{A}_{nk}A_{jk}\Big].\end{aligned} \tag{3.3.98}$$

Now we can express (3.3.87) and (3.3.88) explicitly by substituting $\mathscr{L}_0$ into them. Remember that those two equations were written for the kth particle. Taking the partials appearing in (3.3.87) and (3.3.88), we have

$$\frac{\partial\mathscr{L}_0}{\partial\dot{\phi}_k} = 1 \tag{3.3.99}$$

and

$$\frac{\partial\mathscr{L}_0}{\partial\dot{A}_{jk}} = \left[\rho\dot{A}_{jk} - \rho\alpha_{jj}A_{jk} - \lambda(\hat{t}\cdot\vec{\nabla}\alpha_{jj})A_{jk}\right] \tag{3.3.100}$$

where

$$\frac{\partial}{\partial\dot{A}_{jk}}\left(\sum_{n=1}^{3}\alpha_{jn}\dot{A}_{nk}A_{jk}\right) = \alpha_{jj}A_{jk}.$$

Now we substitute (3.3.99) and (3.3.100) into (3.3.87) and (3.3.88). Equations (3.3.87) and (3.3.88) are then substituted into equation (3.1.23) which directly gives (3.1.24). Equation (3.1.24), being exactly what we were after, is then written on the right hand side of (3.1.17) and (3.1.18) to give us the final form of these equations of motion. Therefore, our equations of motion of the radial and polar coordinates of the kth discrete mass subjected to hydrodynamic forces are

$$\begin{aligned}
&\rho_k \ddot{r}_k + \rho_k g \cos\theta_k - \rho_k r_k \dot{\theta}_k^2 \\
&\quad = \cos\theta_k \int_{-\infty}^{\infty} d\tau\, \delta(\tau - t) \left\{ [-\eta(\vec{\nabla} v_n) + p\hat{n} + \rho\vec{v}(\vec{v}\cdot\hat{n})] \cdot \hat{e}_3 \right\}_k \\
&\qquad + \varpi_r(r_k, \theta_k) \int_t^{\infty} d\tau\, \Delta a_k v_k^{inc}(\vec{x}, \tau) \cos\vartheta_k \\
&\qquad \times \left[\left| \vec{\Pi}_k^c(\vec{x}, \tau) \cdot \hat{y}' \right| + \alpha \left| \vec{\Pi}_k^{c'}(\vec{x}, \tau) \cdot \hat{y}' \right| \right]
\end{aligned} \tag{3.3.101}$$

and

$$\begin{aligned}
&\rho_k r_k^2 \ddot{\theta}_k - \rho_k g r_k \sin\theta_k + 2\rho_k r_k \dot{r}_k \dot{\theta}_k \\
&\quad = -r_k \sin\theta_k \int_{-\infty}^{\infty} d\tau\, \delta(\tau - t) \left\{ [-\eta(\vec{\nabla} v_n) + p\hat{n} + \rho\vec{v}(\vec{v}\cdot\hat{n})] \cdot \hat{e}_3 \right\}_k \\
&\qquad + \varpi_\theta(r_k, \theta_k) \int_t^{\infty} d\tau\, \Delta a_k v_k^{inc}(\vec{x}, \tau) \cos\vartheta_k \\
&\qquad \times \left[\left| \vec{\Pi}_k^c(\vec{x}, \tau) \cdot \hat{y}' \right| + \alpha \left| \vec{\Pi}_k^{c'}(\vec{x}, \tau) \cdot \hat{y}' \right| \right],
\end{aligned} \tag{3.3.102}$$

respectively, where

$$\begin{aligned}
\left| \vec{\Pi}_k^c(\vec{x}, t) \cdot \hat{y}' \right| &= \left| \sum_{j=1}^{3} \left\{ a_j + \rho \dot{A}_{jk}(\vec{x}, t) - \rho\alpha_{jj} A_{jk}(\vec{x}, t) \right. \right. \\
&\qquad \left. \left. - \lambda(\hat{t} \cdot \vec{\nabla}\alpha_{jj}) A_{jk}(\vec{x}, t) \right\} (\hat{e}_j \cdot \hat{y}') \right|
\end{aligned} \tag{3.3.103}$$

and

$$\left|\vec{\Pi}_k^{c'}(\vec{x},t)\cdot\hat{y}'\right| = \left|\sum_{j=1}^{3}\Big\{a'_j + \rho\dot{A}'_{jk}(\vec{x},t) - \rho\alpha_{jj}A'_{jk}(\vec{x},t) - \lambda(\hat{t}\cdot\vec{\nabla}\alpha_{jj})A'_{jk}(\vec{x},t)\Big\}(\hat{e}_j\cdot\hat{y}')\right| \quad (3.3.104)$$

with $\rho_k = m_k/\Delta V_k$, and a'_j being the direction cosine of the reflected scalar potential canonical momentum density with the inward normal of phase II, $\hat{y}'$. Notice that only the diagonal elements of the α matrix are playing an important role in the dynamics.

The left hand side of (3.3.101) and (3.3.102) are the coordinates of the kth particle and the forces generated due to their motion, for example, the centripetal force in the polar equation (3.3.102). The right hand side, however, includes the forces delivered to the kth particle by the streamline and the forces on the kth particle due to the fluids motion in phase II. Equations (3.3.103) and (3.3.104) are, respectively, the components along the inward normal of phase II ($\hat{y}'$) of the incident and reflected momentum density. Note the generality of our second force term on the right hand side of (3.3.101) and (3.3.102). Any Lagrangian density suggested can be used to get the equations corresponding to (3.3.103) and (3.3.104) that are used in (3.3.102) and (3.3.101) as force terms. This shows the flexibility of our formalism with respect to modeling by means of a new Lagrangian density or by adding an interaction Lagrangian density to the original one. Now we would like to get equations that correspond to continuous rather than discrete mass distributions. This will lead us to a new beam model that we call the water beam model.

3.4 The water beam model

We have now completed the problem of determining the equations of motion for dust particles subject to hydrodynamic forces. These are equations (3.3.101) and (3.3.102). But previously we had said that we would rewrite (3.3.101) and (3.3.102) in a form which applies to a continuous mass distribution. In other words, instead of writing a radial and a polar equation for each discrete mass, we want to write a radial and a polar equation for a continuous distribution of mass. Although this step sounds straightforward it will, as we shall see, lead to many complications. Let us do that next.

First, we note that the superposition principle is assumed so that equations (3.3.101) and (3.3.102) are written as

$$\begin{aligned}
&\sum_{k=1}^{N} \rho_k \ddot{r}_k + \sum_{k=1}^{N} \rho_k g \cos\theta_k - \sum_{k=1}^{N} \rho_k r_k \dot{\theta}_k^2 \\
&= \sum_{k=1}^{N} \cos\theta_k \int_{-\infty}^{\infty} d\tau\, \delta(\tau - t) \left\{ \left[-\eta(\vec{\nabla} v_n) + p\hat{n} + \rho\vec{v}(\vec{v}\cdot\hat{n}) \right] \cdot \hat{e}_3 \right\}_k \\
&\quad + \sum_{k=1}^{N} \varpi_r(r_k, \theta_k) \int_t^{\infty} d\tau\, \Delta a_k v_k^{inc}(\vec{x}, \tau) \cos\vartheta_k \\
&\quad \times \left[\left| \vec{\Pi}_k^c(\vec{x}, \tau) \cdot \hat{y}' \right| + \alpha \left| \vec{\Pi}_k^{c'}(\vec{x}, \tau) \cdot \hat{y}' \right| \right] d\tau
\end{aligned} \tag{3.4.105}$$

and

$$\begin{aligned}
&\sum_{k=1}^{N} \rho_k r_k^2 \ddot{\theta}_k - \sum_{k=1}^{N} \rho_k g r_k \sin\theta_k + 2\sum_{k=1}^{N} \rho_k r_k \dot{r}_k \dot{\theta}_k \\
&= -\sum_{k=1}^{N} r_k \sin\theta_k \int_{-\infty}^{\infty} d\tau\, \delta(\tau - t) \left\{ \left[-\eta(\vec{\nabla} v_n) + p\hat{n} + \rho\vec{v}(\vec{v}\cdot\hat{n}) \right] \cdot \hat{e}_3 \right\}_k \\
&\quad + \sum_{k=1}^{N} \varpi_\theta(r_k, \theta_k) \int_t^{\infty} d\tau\, \Delta a_k v_k^{inc}(\vec{x}, \tau) \cos\vartheta_k \\
&\quad \times \left[\left| \vec{\Pi}_k^c(\vec{x}, \tau) \cdot \hat{y}' \right| + \alpha \left| \vec{\Pi}_k^{c'}(\vec{x}, \tau) \cdot \hat{y}' \right| \right] d\tau,
\end{aligned} \tag{3.4.106}$$

that is, we are summing the right hand side and the left hand side of equations (3.3.101) and (3.3.102) over all particles and assume that we have N such particles and streamlines. In the following pages we will introduce what we call the "water beam" model. This model will bring in some interactions that will make the application of the principle of superposition impossible. However, we will resolve this problem in our formalism of what we called "the interactions of interferences".

To take (3.4.105) and (3.4.106) to a continuous limit we take $N >> 1$ and introduce the following transformations from a discrete to a continuous limit

$$\begin{aligned}
\sum_k \rho_k \Delta V_k &\rightarrow \int d^3x\, \rho(\vec{x},t) \\
q_k(t) &\rightarrow q(\vec{x},t) \\
v_k^{inc}(\vec{x},t) &\rightarrow \left[v^{inc}(\vec{x},t)\right](y,z) \\
\vec{\Pi}_k^c(\vec{x},t) &\rightarrow \left[\vec{\Pi}^c(\vec{x},t)\right](y,z) \\
\vec{\Pi}_k^{c'}(\vec{x},t) &\rightarrow \left[\vec{\Pi}^{c'}(\vec{x},t)\right](y,z)
\end{aligned}$$

In the above transformations q_k is the generalized coordinate of the kth particle or streamline. Note that as a result of this transformation the generalized coordinate is a function of $\vec{x}$ and t, that is, it is a field, which means that the total time derivative should be expressed in terms of what is known as the "material derivative". We will consider this later. For now, we use the transformations above to write the left hand side of (3.4.105) in its continuous form as

$$\int d^3x \rho(\vec{x},t)\frac{d^2r(\vec{x},t)}{dt^2} + \int d^3x \rho(\vec{x},t) g\cos[\theta(\vec{x},t)] - \int d^3x \rho(\vec{x},t) r(\vec{x},t)\left[\frac{d\theta(\vec{x},t)}{dt}\right]^2 \tag{3.4.107}$$

and that of (3.4.106) as

$$\int d^3x\rho(\vec{x},t)[r(\vec{x},t)]^2\frac{d^2\theta(\vec{x},t)}{dt^2} - \int d^3x\rho(\vec{x},t)gr(\vec{x},t)\sin[\theta(\vec{x},t)] + 2\int d^3x\rho(\vec{x},t)r(\vec{x},t)\frac{dr(\vec{x},t)}{dt}\frac{d\theta(\vec{x},t)}{dt}. \tag{3.4.108}$$

On the other hand, the right hand side of both (3.4.105) and (3.4.106) have in them the corresponding components of

$$\begin{aligned}
&\int_{-\infty}^{\infty} dy \int_{-\infty}^{\infty} dx \left(\cos\theta(x,y,t), -r(x,y,t)\sin\theta(x,y,t)\right) \int_{-\infty}^{\infty} d\tau\, \delta(\tau - t) \\
&\quad\times \left\{[-\eta(\vec{\nabla}v_n) + p\hat{n} + \rho\vec{v}(\vec{v}\cdot\hat{n})]\cdot\hat{e}_3\right\}(x,y) \\
&\quad+ \int_0^{z_0} dz \int_{-\infty}^{\infty} dx \left(\varpi_r\left(r(x,z,t),\theta(x,z,t)\right), \varpi_\theta\left(r(x,z,t),\theta(x,z,t)\right)\right) \\
&\quad\times \int_t^{\infty} d\tau\, da(x,z)[v^{inc}(\vec{x},\tau)](x,z)\cos[\vartheta(x,z)] \\
&\quad\times \left[\left|[\vec{\Pi}^c(\vec{x},\tau)\cdot\hat{y}'](x,z)\right| + \alpha\left|[\vec{\Pi}^{c'}(\vec{x},\tau)\cdot\hat{y}'](x,z)\right|\right]
\end{aligned} \tag{3.4.109}$$

where z_0 is the height of the liquid water. In equation (3.4.109) the integration variables x, y, and z that are introduced as a result of transforming (3.4.105) and (3.4.106) to a continuous limit are entirely independent of the variables already appearing in (3.4.105) and (3.4.106) as variables of functions, namely $\vec{x}$ and τ. Using the same symbol for two independent variables could occur and it would be misleading. Therefore we will suppress $\vec{x}$ and τ in (3.4.109) and write the dynamical variables as functions of the new continuous variables only. So we write (3.4.109) as

$$\begin{aligned}
&\int_{-\infty}^{\infty} dy \int_{-\infty}^{\infty} dx \left(\cos\theta(x,y,t), -r(x,y,t)\sin\theta(x,y,t)\right) \int_{-\infty}^{\infty} d\tau\, \delta(\tau - t) \\
&\quad\times \left\{[-\eta(\vec{\nabla}v_n) + p\hat{n} + \rho\vec{v}(\vec{v}\cdot\hat{n})]\cdot\hat{e}_3\right\}(x,y)
\end{aligned}$$

$$
\begin{aligned}
&+\int_0^{z_0} dz \int_{-\infty}^{\infty} dx \left(\varpi_r\big(r(x,z,t),\theta(x,z,t)\big), \varpi_\theta\big(r(x,z,t),\theta(x,z,t)\big)\right) \\
&\quad\times \int_t^{\infty} d\tau \, da(x,z) v^{inc}(x,z) \cos[\vartheta(x,z)] \\
&\quad\times \left[\left|[\vec{\Pi}^c \cdot \hat{y}'](x,z)\right| + \alpha \left|[\vec{\Pi}^{c'} \cdot \hat{y}'](x,z)\right|\right]
\end{aligned}
\tag{3.4.110}
$$

so that (3.4.110) is similar to (3.4.109) except the variables $\vec{x}$ and τ do not appear in equation (3.4.110) as variables of dynamical functions. Recall that in equation (3.1.23), Δa_i corresponded to the finite area in the plane dividing the phases I and II at which the ith streamline would arrive with velocity v_i. In our continuous formalism we changed the discrete index to continuous variables and fixed $\vec{x}$. Now, in order for us to consider the ith (kth) streamline arriving at all possible finite areas of the plane dividing phases I and II, we integrate (3.4.110) over the plane dividing the two phases. Likewise, if we want to include all the possible angles of incidence of the kth streamline in phase II, we integrate the angle of incidence over the interval $[-\pi/2, \pi/2]$.

Adding the changes mentioned above to equation (3.4.110) we can then write

$$
\begin{aligned}
&\int_{-\infty}^{\infty} dy \int_{-\infty}^{\infty} dx \left(\cos\theta(x,y,t), -r(x,y,t)\sin\theta(x,y,t)\right) \int_{-\infty}^{\infty} d\tau \, \delta(\tau - t) \\
&\quad\times \left\{[-\eta(\vec{\nabla} v_n) + p\hat{n} + \rho\vec{v}(\vec{v}\cdot\hat{n})]\cdot\hat{e}_3\right\}(x,y) \\
&+\int_S \int_0^{z_0} dz \int_{-\infty}^{\infty} dx \left(\varpi_r\big(r(x,z,t),\theta(x,z,t)\big), \varpi_\theta\big(r(x,z,t),\theta(x,z,t)\big)\right) \\
&\quad\times \int_t^{\infty} d\tau \, da(x,z) \int_{-\pi/2}^{\pi/2} d\vartheta(x,z) v^{inc}(x,z) \cos[\vartheta(x,z)] \\
&\quad\times \left[\left|[\vec{\Pi}^c \cdot \hat{y}'](x,z)\right| + \alpha \left|[\vec{\Pi}^{c'} \cdot \hat{y}'](x,z)\right|\right],
\end{aligned}
\tag{3.4.111}
$$

where S is the surface area of the plane dividing phases I and II and where the integral over ϑ from $-\pi/2$ to $\pi/2$ is taken over all angles of incidence. From

(3.4.107), (3.4.108), and (3.4.111) we have

$$
\begin{aligned}
\int d^3x\, \rho(\vec{x},t)\ddot{r}(\vec{x},t) \;&+\; \int d^3x \rho(\vec{x},t) g \cos[\theta(\vec{x},t)] - \int d^3x \rho(\vec{x},t) r(\vec{x},t)\, \left[\dot{\theta}(\vec{x},t)\right]^2 \\
&= \int_{-\infty}^{\infty} dy \int_{-\infty}^{\infty} dx\, \cos\theta(x,y,t) \int_{-\infty}^{\infty} d\tau\, \delta(\tau - t) \\
&\quad \times \left\{[-\eta(\vec{\nabla} v_n) + p\hat{n} + \rho\vec{v}(\vec{v}\cdot\hat{n})]\cdot\hat{e}_3\right\}(x,y) \\
&\quad + \int_S \int_0^{z_0} dz \int_{-\infty}^{\infty} dx\, \varpi_r\Big(r(x,z,t), \theta(x,z,t)\Big) \\
&\quad \times \int_t^{\infty} d\tau\, da(x,z) \int_{-\pi/2}^{\pi/2} d\vartheta(x,z) v^{inc}(x,z) \cos[\vartheta(x,z)] \\
&\quad \times \left[\left|[\vec{\Pi}^c \cdot \hat{y}'](x,z)\right| + \alpha \left|[\vec{\Pi}^{c'} \cdot \hat{y}'](x,z)\right|\right], \qquad (3.4.112)
\end{aligned}
$$

and

$$
\begin{aligned}
&\int d^3x\, \rho(\vec{x},t)[r(\vec{x},t)]^2 \ddot{\theta}(\vec{x},t) - \int d^3x \rho(\vec{x},t) g r(\vec{x},t) \sin[\theta(\vec{x},t)] \\
&\quad + 2\int d^3x \rho(\vec{x},t) r(\vec{x},t) \dot{r}(\vec{x},t) \dot{\theta}(\vec{x},t) \\
&= -\int_{-\infty}^{\infty} dy \int_{-\infty}^{\infty} dx\, r(x,y,t) \sin\theta(x,y,t) \int_{-\infty}^{\infty} d\tau\, \delta(\tau - t) \\
&\quad \times \left\{[-\eta(\vec{\nabla} v_n) + p\hat{n} + \rho\vec{v}(\vec{v}\cdot\hat{n})]\cdot\hat{e}_3\right\}(x,y) \\
&\quad + \int_S \int_0^{z_0} dz \int_{-\infty}^{\infty} dx\, \varpi_\theta\Big(r(x,z,t), \theta(x,z,t)\Big) \\
&\quad \times \int_t^{\infty} d\tau\, da(x,z) \int_{-\pi/2}^{\pi/2} d\vartheta(x,z) v^{inc}(x,z) \cos[\vartheta(x,z)] \\
&\quad \times \left[\left|[\vec{\Pi}^c \cdot \hat{y}'](x,z)\right| + \alpha \left|[\vec{\Pi}^{c'} \cdot \hat{y}'](x,z)\right|\right] \qquad (3.4.113)
\end{aligned}
$$

as the continuous limits of (3.4.105) and (3.4.106) (obtained from (3.3.101) and (3.3.102)). Figure (6) below demonstrates the dynamics pertaining to equations (3.4.112) and (3.4.113).

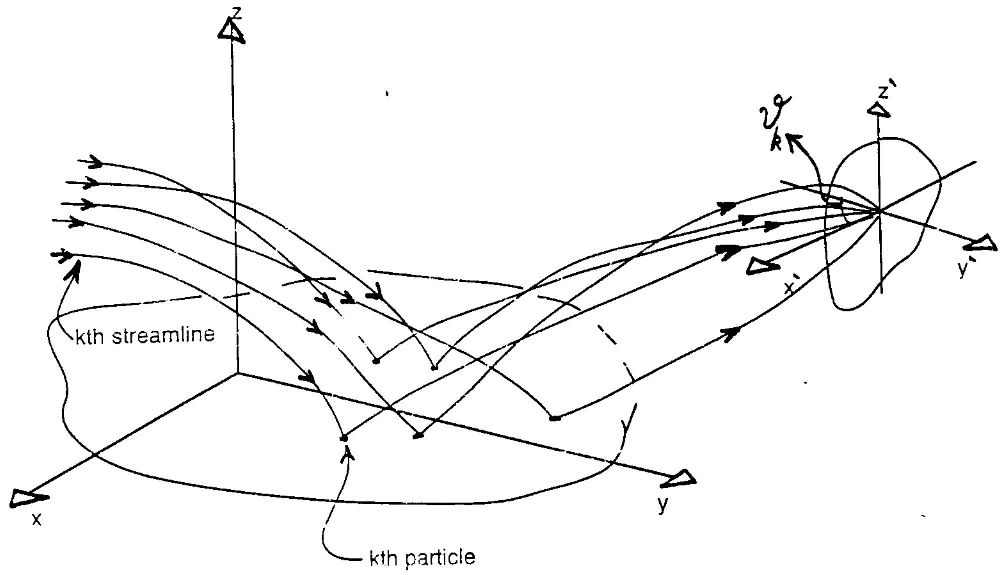

Figure 6: Streamlines scattering with residing particles.

We will finally derive the equation of motion of the azimuth in addition to the radial and the polar equations given in (3.4.112) and (3.4.113). To simplify the writing, we introduce the following two definitions.

$$\begin{aligned}\Big(F_r^1(\vec{x},t),F_\theta^1(\vec{x},t)\Big) \;\equiv\; & \int_{-\infty}^{\infty} dy \int_{-\infty}^{\infty} dx \,\Big(\cos\theta(x,y,t), -r(x,y,t)\sin\theta(x,y,t)\Big) \\ & \times \int_{-\infty}^{\infty} d\tau\, \delta(\tau - t) \\ & \times \Big\{[-\eta(\vec{\nabla} v_n) + p\hat{n} + \rho\vec{v}(\vec{v}\cdot\hat{n})]\cdot\hat{e}_3\Big\}(x,y) \quad (3.4.114)\end{aligned}$$

and

$$\begin{aligned}\Big(F_r^2(\vec{x},t),F_\theta^2(\vec{x},t)\Big) \;\equiv\; & \int_S \int_0^{z_0} dz \int_{-\infty}^{\infty} dx \\ & \times \Big(\varpi_r\Big(r(x,z,t),\theta(x,z,t)\Big), \varpi_\theta\Big(r(x,z,t),\theta(x,z,t)\Big)\Big) \\ & \times \int_t^{\infty} d\tau\, da(x,z) \int_{-\pi/2}^{\pi/2} d\vartheta(x,z)\, v^{inc}(x,z)\cos[\vartheta(x,z)] \\ & \times \Big[\Big|[\vec{\Pi}^c\cdot\hat{y}'](x,z)\Big| + \alpha\Big|[\vec{\Pi}^{c'}\cdot\hat{y}'](x,z)\Big|\Big], \qquad (3.4.115)\end{aligned}$$

which are the right hand side forces of equations (3.4.112) and (3.4.113). Let us now write the equation of motion for the azimuth.

According to equations (3.1.1), (3.1.2), and (3.1.3) we have the following for the azimuth

$$\frac{d}{dt}\left(\frac{\partial T_i}{\partial \dot{\varphi}_i}\right) - \frac{\partial T_i}{\partial \varphi_i} = -m_i g \frac{\partial z_i}{\partial \varphi_i} + f^a_{\varphi_i} + f^b_{\varphi_i} \tag{3.4.116}$$

where φ_i is the azimuthal component of the ith particle. In equation (3.4.116) we have $T_i = \frac{1}{2}m_i(\dot{r}_i^2 + r_i^2\dot{\theta}_i^2 + r_i^2 \sin\theta_i \dot{\varphi}_i^2)$. This will change the form of the radial and the polar equations as well, so we write those as well. Applying (3.1.2) to this new T_i the radial, the polar, and the azimuthal equations of motion are, respectively,

$$\begin{aligned}
\frac{d}{dt}\left(\frac{\partial T_i}{\partial \dot{r}_i}\right) - \frac{\partial T_i}{\partial r_i} &= -m_i g \frac{\partial z_i}{\partial r_i} + f^a_{r_i} + f^b_{r_i} \\
\frac{d}{dt}\left(\frac{\partial T_i}{\partial \dot{\theta}_i}\right) - \frac{\partial T_i}{\partial \theta_i} &= -m_i g \frac{\partial z_i}{\partial \theta_i} + f^a_{\theta_i} + f^b_{\theta_i} \\
\frac{d}{dt}\left(\frac{\partial T_i}{\partial \dot{\varphi}_i}\right) - \frac{\partial T_i}{\partial \varphi_i} &= -m_i g \frac{\partial z_i}{\partial \varphi_i} + f^a_{\varphi_i} + f^b_{\varphi_i},
\end{aligned}$$

which after dividing by ΔV_i and reinterpreting the forces as force densities (using the same notation) give

$$\begin{aligned}
\rho_i\ddot{r}_i - \rho_i r_i\dot{\theta}_i^2 - \rho_i r_i \sin\theta_i\dot{\varphi}_i^2 + \rho_i g\cos\theta_i &= f^a_{r_i} + f^b_{r_i} \\
\rho_i r_i^2\ddot{\theta}_i + 2\rho_i r_i\dot{r}_i\dot{\theta}_i - \frac{1}{2}\rho_i r_i^2\cos\theta_i\dot{\varphi}_i^2 - \rho_i g r_i\sin\theta_i &= f^a_{\theta_i} + f^b_{\theta_i} \\
\rho_i r_i^2\sin\theta_i\ddot{\varphi}_i + \rho_i r_i^2\cos\theta_i\dot{\theta}_i\dot{\varphi}_i + 2\rho_i r_i\dot{r}_i\sin\theta_i\dot{\varphi}_i &= f^a_{\varphi_i} + f^b_{\varphi_i}
\end{aligned}$$

as the discrete form of the radial, the polar, and the azimuthal equations of motion, respectively. Taking the above equations to a continuous limit, we get

$$\begin{aligned}
\int d^3x\, \rho(\vec{x},t)\frac{dv_r(\vec{x},t)}{dt} &+ \int d^3x\, g\rho(\vec{x},t)\cos[\theta(\vec{x},t)] - \int d^3x\, \rho(\vec{x},t)r(\vec{x},t)v_\theta^2(\vec{x},t) \\
&- \int d^3x\, \rho(\vec{x},t)r(\vec{x},t)\sin[\theta(\vec{x},t)]v_\varphi^2(\vec{x},t) \\
&= F_r^1(\vec{x},t) + F_r^2(\vec{x},t),
\end{aligned} \tag{3.4.117}$$

$$
\begin{aligned}
\int d^3x\, \rho(\vec{x},t) r^2(\vec{x},t) \frac{dv_\theta(\vec{x},t)}{dt} \quad &- \int d^3x\, \rho(\vec{x},t) g r(\vec{x},t) \sin[\theta(\vec{x},t)] \\
&+ 2\int d^3x\, \rho(\vec{x},t) r(\vec{x},t) v_r(\vec{x},t) v_\theta(\vec{x},t) \\
&+ \int d^3x\, \rho(\vec{x},t) r^2(\vec{x},t) \cos[\theta(\vec{x},t)] v_\varphi^2(\vec{x},t) \\
&= F_\theta^1(\vec{x},t) + F_\theta^2(\vec{x},t), \qquad (3.4.118)
\end{aligned}
$$

and

$$
\begin{aligned}
&\int d^3x\, \rho(\vec{x},t) r^2(\vec{x},t) \sin[\theta(\vec{x},t)] \frac{dv_\varphi(\vec{x},t)}{dt} \\
&\quad + \int d^3x\, \rho(\vec{x},t) g r^2(\vec{x},t) \cos[\theta(\vec{x},t)] v_\theta(\vec{x},t) v_\varphi(\vec{x},t) \\
&\quad + 2\int d^3x\, \rho(\vec{x},t) r(\vec{x},t) v_r(\vec{x},t) \sin[\theta(\vec{x},t)] v_\varphi(\vec{x},t) \\
&= F_\varphi^1(\vec{x},t) + F_\varphi^2(\vec{x},t), \qquad (3.4.119)
\end{aligned}
$$

where

$$
\begin{aligned}
\Big(F_r^1(\vec{x},t), F_\theta^1(\vec{x},t), F_\varphi^1(\vec{x},t)\Big) \quad &\equiv \int_{-\infty}^{\infty} dy \int_{-\infty}^{\infty} dx \\
&\times \Big(\cos\theta(x,y,t), -r(x,y,t)\sin\theta(x,y,t), 0\Big) \\
&\times \int_{-\infty}^{\infty} d\tau\, \delta(\tau - t) \\
&\times \Big\{[-\eta(\vec{\nabla} v_n) + p\hat{n} + \rho\vec{v}(\vec{v}\cdot\hat{n})]\cdot\hat{e}_3\Big\}(x,y)
\end{aligned}
$$

and

$$
\begin{aligned}
\Big(F_r^2(\vec{x},t), F_\theta^2(\vec{x},t), F_\varphi^2(\vec{x},t)\Big) \equiv \int_0^{z_0} dz \int_{-\infty}^{\infty} dx\, \Big(&\varpi_r\Big(r(x,z,t),\theta(x,z,t),\varphi(x,z,t)\Big), \\
\varpi_\theta\Big(r(x,z,t),\theta(x,z,t),\varphi(x,z,t)\Big), &\varpi_\varphi\Big(r(x,z,t),\theta(x,z,t),\varphi(x,z,t)\Big)\Big) \\
\times \int_t^{\infty} d\tau \int_S da(x,z) \int_{-\pi/2}^{\pi/2} &d\vartheta(x,z)\, v^{inc}(x,z)\cos[\vartheta(x,z)]
\end{aligned}
$$

$$\times \left[\left| [\vec{\Pi}^{c} \cdot \hat{y}'](x,z) \right| + \alpha \left| [\vec{\Pi}^{c'} \cdot \hat{y}'](x,z) \right| \right],$$

where

$$\begin{aligned}
\varpi_r(r,\theta,\varphi) &\equiv \sin\theta\cos\varphi + \sin\theta\sin\varphi + \cos\theta \\
\varpi_\theta(r,\theta,\varphi) &\equiv r\cos\theta\cos\varphi + r\cos\theta\sin\varphi - r\sin\theta \\
\varpi_\varphi(r,\theta,\varphi) &\equiv -r\sin\theta\sin\varphi + r\sin\theta\cos\varphi.
\end{aligned}$$

We also made the following identifications in the above equations $dr(\vec{x},t)/dt \equiv v_r(\vec{x},t)$, $d\theta(\vec{x},t)/dt \equiv v_\theta(\vec{x},t)$ and $d\varphi(\vec{x},t)/dt \equiv v_\varphi(\vec{x},t)$. Equations (3.4.117), (3.4.118), and (3.4.119) are, respectively, the radial, the polar, and the azimuthal equations of motion in continuous form. The components of $\vec{F}^1(\vec{x},t)$ and $\vec{F}^2(\vec{x},t)$ appearing in (3.4.114) and (3.4.115), respectively, are the hydrodynamic forces on the particles. The next step is to write (3.4.117), (3.4.118) and (3.4.119) in terms of the material derivative. Note that for any function of position and time, for example $\vec{v}(\vec{x},t)$, the material derivative is written as

$$\frac{d\vec{v}(\vec{x},t)}{dt} = \frac{\partial\vec{v}(\vec{x},t)}{\partial t} + (\vec{v}\cdot\vec{\nabla})\vec{v}.$$

In spherical coordinates it becomes

$$\frac{d\vec{v}}{dt} = \frac{\partial\vec{v}(\vec{x},t)}{\partial t} + \left(v_r\frac{\partial}{\partial r} + \frac{v_\theta}{r}\frac{\partial}{\partial\theta} + \frac{v_\varphi}{r\sin\theta}\frac{\partial}{\partial\varphi} \right)\vec{v}$$

where $\vec{v} = v_r\hat{r} + v_\theta\hat{\theta} + v_\varphi\hat{\varphi}$, which in turn leads us to write

$$\frac{d\vec{v}(\vec{x},t)}{dt} = \frac{d}{dt}\left(v_r\hat{r} + v_\theta\hat{\theta} + v_\varphi\hat{\varphi}\right) = \frac{\partial}{\partial t}\left(v_r\hat{r} + v_\theta\hat{\theta} + v_\varphi\hat{\varphi}\right) + v_r\frac{\partial\vec{v}}{\partial r} + \frac{v_\theta}{r}\frac{\partial\vec{v}}{\partial\theta} + \frac{v_\varphi}{r\sin\theta}\frac{\partial\vec{v}}{\partial\varphi} \tag{3.4.120}$$

By equating the corresponding components of the derivatives we can find the material derivative in the radial, polar and azimuthal components. The quantities $\partial\vec{v}/\partial r$. $\partial\vec{v}/\partial\theta$, and $\partial\vec{v}/\partial\varphi$ should be calculated separately. We then have

$$\begin{aligned}
\frac{\partial\vec{v}}{\partial r} &= \frac{\partial v_r}{\partial r}\hat{r} + \frac{\partial v_\theta}{\partial r}\hat{\theta} + \frac{\partial v_\varphi}{\partial r}\hat{\varphi} \\
\frac{\partial\hat{r}}{\partial r} &= \frac{\partial\hat{\theta}}{\partial r} = \frac{\partial\hat{\varphi}}{\partial r} = 0 \\
\frac{\partial\vec{v}}{\partial\theta} &= \frac{\partial v_r}{\partial\theta}\hat{r} + v_r\frac{\partial\hat{r}}{\partial\theta} + \frac{\partial v_\theta}{\partial\theta}\hat{\theta} + v_\theta\frac{\partial\hat{\theta}}{\partial\theta} + \frac{\partial v_\varphi}{\partial\theta}\hat{\varphi} + v_\varphi\frac{\partial\hat{\varphi}}{\partial\theta}.
\end{aligned}$$

But

$$\frac{\partial\hat{r}}{\partial\theta} = \hat{\theta}, \qquad \frac{\partial\hat{\theta}}{\partial\theta} = -\hat{r}, \qquad \frac{\partial\hat{\varphi}}{\partial\theta} = 0,$$

so that

$$\frac{\partial\vec{v}}{\partial\theta} = \left(\frac{\partial v_r}{\partial\theta} - v_\theta\right)\hat{r} + \left(v_r + \frac{\partial v_\theta}{\partial\theta}\right)\hat{\theta} + \frac{\partial v_\varphi}{\partial\theta}\hat{\varphi},$$

and

$$\frac{\partial\vec{v}}{\partial\varphi} = \left(\frac{\partial v_r}{\partial\varphi} - v_\varphi\sin\theta\right)\hat{r} + \left(\frac{\partial v_\theta}{\partial\varphi} - v_\varphi\cos\theta\right)\hat{\theta} + \left(v_r\sin\varphi + v_\theta\cos\theta + \frac{\partial v_\varphi}{\partial\varphi}\right)\hat{\varphi}.$$

The implicit time differentiation on the right hand side of $d\vec{v}/dt$ should also be calculated separately. We have then

$$\begin{aligned}
\frac{\partial}{\partial t}\left(v_r\hat{r} + v_\theta\hat{\theta} + v_\varphi\hat{\varphi}\right) &= \frac{\partial(v_r\hat{r})}{\partial t} + \frac{\partial(v_\theta\hat{\theta})}{\partial t} + \frac{\partial(v_\varphi\hat{\varphi})}{\partial t} \\
\frac{\partial(v_r\hat{r})}{\partial t} &= \frac{\partial v_r}{\partial t}\hat{r} + v_r\frac{\partial\hat{r}}{\partial t}.
\end{aligned}$$

But $\partial\hat{r}/\partial t = \dot{\theta}\hat{\theta} + \dot{\varphi}\sin\theta\hat{\varphi} = v_\theta\hat{\theta} + v_\varphi\sin\theta\hat{\varphi}$; hence,

$$\frac{\partial(v_r\hat{r})}{\partial t} = \frac{\partial v_r}{\partial t}\hat{r} + v_r v_\theta\hat{\theta} + v_r v_\varphi\sin\theta\hat{\varphi}$$

$$\frac{\partial(v_\theta\hat{\theta})}{\partial t} = \frac{\partial v_\theta}{\partial t}\hat{\theta} + v_\theta\frac{\partial\hat{\theta}}{\partial t}.$$

Since $\partial\hat{\theta}/\partial t = -v_\theta\hat{r} + v_\varphi\cos\theta\hat{\varphi}$,

$$\begin{aligned}\frac{\partial(v_\theta\hat{\theta})}{\partial t} &= -v_\theta^2\hat{r} + \frac{\partial v_\theta}{\partial t}\hat{\theta} + v_\theta v_\varphi\cos\theta\hat{\varphi}\\ \frac{\partial(v_\varphi\hat{\varphi})}{\partial t} &= \frac{\partial v_\varphi}{\partial t}\hat{\varphi} + v_\varphi\frac{\partial\hat{\varphi}}{\partial t}.\end{aligned}$$

Finally, $\partial\hat{\varphi}/\partial t = -v_\varphi\sin\theta\hat{r} - v_\varphi\cos\theta\hat{\theta}$, which gives

$$\frac{\partial(v_\varphi\hat{\varphi})}{\partial t} = -v_\varphi^2\sin\theta\hat{r} - v_\varphi^2\cos\theta\hat{\theta} + \frac{\partial v_\varphi}{\partial t}\hat{\varphi}.$$

We now can rewrite equation (3.4.120) in its explicit form, which will be

$$\begin{aligned}\frac{d\vec{v}(\vec{x},t)}{dt} = {} & \frac{\partial v_r}{\partial t}\hat{r} + v_r v_\theta\hat{\theta} + v_r v_\varphi\sin\theta\hat{\varphi} - v_\theta^2\hat{r} + \frac{\partial v_\theta}{\partial t}\hat{\theta}\\ & + v_\theta v_\varphi\cos\theta\hat{\varphi} - v_\varphi^2\sin\theta\hat{r} - v_\varphi^2\cos\theta\hat{\theta} + \frac{\partial v_\varphi}{\partial t}\hat{\varphi} + v_r\frac{\partial v_r}{\partial r}\hat{r}\\ & + v_r\frac{\partial v_\theta}{\partial r}\hat{\theta} + v_r\frac{\partial v_\varphi}{\partial r}\hat{\varphi} + \frac{v_\theta}{r}\left(\frac{\partial v_r}{\partial\theta} - v_\theta\right)\hat{r} + \frac{v_\theta}{r}\left(v_r + \frac{\partial v_\theta}{\partial\theta}\right)\hat{\theta}\\ & + \frac{v_\theta}{r}\frac{\partial v_\varphi}{\partial\theta}\hat{\varphi} + \frac{v_\varphi}{r\sin\theta}\left(\frac{\partial v_r}{\partial\varphi} - v_\varphi\sin\theta\right)\hat{r}\\ & + \frac{v_\varphi}{r\sin\theta}\left(\frac{\partial v_\theta}{\partial\varphi} - v_\varphi\cos\theta\right)\hat{\theta} + \frac{v_\varphi}{r\sin\theta}\left(v_r\sin\varphi + v_\theta\cos\theta + \frac{\partial v_\varphi}{\partial\varphi}\right)\hat{\varphi}.\end{aligned}$$

After equating the corresponding components, we find that

$$\begin{aligned}\frac{d}{dt}(v_r, v_\theta, v_\varphi) = {} & \left[\frac{\partial v_r}{\partial t} - v_\theta^2 - v_\varphi^2\sin\theta + v_r\frac{\partial v_r}{\partial r}\right.\\ & \left. + \frac{v_\theta}{r}\left(\frac{\partial v_r}{\partial\theta} - v_\theta\right) + \frac{v_\varphi}{r\sin\theta}\left(\frac{\partial v_r}{\partial\varphi} - v_\varphi\sin\theta\right)\right]\hat{r}\\ & + \left[v_r v_\theta + \frac{\partial v_\theta}{\partial t} - v_\varphi^2\cos\theta + v_r\frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r}\left(v_r + \frac{\partial v_\theta}{\partial\theta}\right) + \frac{v_\varphi}{r\sin\theta}\left(\frac{\partial v_\theta}{\partial\varphi} - v_\varphi\cos\theta\right)\right]\hat{\theta}\end{aligned}$$

$$+\left[v_r v_\varphi \sin\theta + v_\theta v_\varphi \cos\theta + \frac{\partial v_\varphi}{\partial t} + v_r \frac{\partial v_\varphi}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_\varphi}{\partial \theta} + \frac{v_\varphi}{r\sin\theta}\left(v_r \sin\varphi + v_\theta \cos\theta + \frac{\partial v_\varphi}{\partial \phi}\right)\right]\hat{\varphi}.$$

Equating the corresponding components of the derivatives, we have for the radial component

$$\frac{dv_r}{dt} = \frac{\partial v_r}{\partial t} - v_\theta^2 - v_\varphi^2 \sin\theta + v_r \frac{\partial v_r}{\partial r} + \frac{v_\theta}{r}\left(\frac{\partial v_r}{\partial \theta} - v_\theta\right) + \frac{v_\varphi}{r\sin\theta}\left(\frac{\partial v_r}{\partial \varphi} - v_\varphi \sin\theta\right), \tag{3.4.121}$$

the polar component

$$\frac{dv_\theta}{dt} = v_r v_\theta + \frac{\partial v_\theta}{\partial t} - v_\varphi^2 \cos\theta + v_r \frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r}\left(v_r + \frac{\partial v_\theta}{\partial \theta}\right) + \frac{v_\varphi}{r\sin\theta}\left(\frac{\partial v_\theta}{\partial \varphi} - v_\varphi \cos\theta\right), \tag{3.4.122}$$

and the azimuthal component

$$\frac{dv_\varphi}{dt} = v_r v_\varphi \sin\theta + v_\theta v_\varphi \cos\theta + \frac{\partial v_\varphi}{\partial t} + v_r \frac{\partial v_\varphi}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_\varphi}{\partial \theta} + \frac{v_\varphi}{r\sin\theta}\left(v_r \sin\varphi + v_\theta \cos\theta + \frac{\partial v_\varphi}{\partial \phi}\right). \tag{3.4.123}$$

Substituting (3.4.121), (3.4.122), and (3.4.123) into (3.4.117), (3.4.118), and (3.4.119) respectively, we obtain the radial component of the equation of motion in the form

$$\begin{aligned}
&\int d^3x\, \rho(\vec{x},t)\frac{\partial v_r(\vec{x},t)}{\partial t} - \int d^3x\, \rho(\vec{x},t)[v_\theta(\vec{x},t)]^2 - \int d^3x\, \rho(\vec{x},t)[v_\varphi(\vec{x},t)]^2 \sin[\theta(\vec{x},t)] \\
&\quad + \int d^3x\, \rho(\vec{x},t) v_r(\vec{x},t)\frac{\partial v_r(\vec{x},t)}{\partial r} + \int d^3x\, \rho(\vec{x},t)\frac{v_\theta(\vec{x},t)}{r}\left[\frac{\partial v_r(\vec{x},t)}{\partial \theta} - v_\theta(\vec{x},t)\right] \\
&\quad + \int d^3x\, \rho(\vec{x},t)\frac{v_\varphi(\vec{x},t)}{r\sin\theta}\left[\frac{\partial v_r(\vec{x},t)}{\partial \varphi} - v_\varphi(\vec{x},t)\sin[\theta(\vec{x},t)]\right] \\
&\quad + \int d^3x\, g\rho(\vec{x},t)\cos[\theta(\vec{x},t)] - \int d^3x\, \rho(\vec{x},t) r(\vec{x},t) v_\theta^2(\vec{x},t) \\
&\quad - \int d^3x\, \rho(\vec{x},t) r(\vec{x},t)\sin[\theta(\vec{x},t)] v_\varphi^2(\vec{x},t) = \sum_{i=1}^{2} F_r^i(\vec{x},t),
\end{aligned} \tag{3.4.124}$$

the polar component as

$$\int d^3x\, \rho(\vec{x},t) r^2(\vec{x},t) \frac{\partial v_\theta(\vec{x},t)}{\partial t}$$
$$+ \int d^3x \rho(\vec{x},t) r^2(\vec{x},t) v_r(\vec{x},t) v_\theta(\vec{x},t) + \int d^3x \rho(\vec{x},t) r^2(\vec{x},t) v_r(\vec{x},t) \frac{\partial v_\theta(\vec{x},t)}{\partial r}$$
$$+ 3 \int d^3x \rho(\vec{x},t) r(\vec{x},t) v_\theta(\vec{x},t) v_r(\vec{x},t) + \int d^3x \rho(\vec{x},t) r(\vec{x},t) v_\theta(\vec{x},t) \frac{\partial v_\theta(\vec{x},t)}{\partial \theta}$$
$$+ \int d^3x \rho(\vec{x},t) r(\vec{x},t) \frac{v_\varphi(\vec{x},t)}{\sin\theta} \frac{\partial v_\theta(\vec{x},t)}{\partial \varphi} + \int d^3x \rho(\vec{x},t) r(\vec{x},t) [v_\varphi(\vec{x},t)]^2 (\cos\theta - \cot\theta)$$
$$- \int d^3x \rho(\vec{x},t) g r(\vec{x},t) \sin\theta = \sum_{i=1}^{2} F_\theta^i(\vec{x},t), \tag{3.4.125}$$

and the azimuthal as

$$\int d^3x\, \rho(\vec{x},t) r^2(\vec{x},t) \sin[\theta(\vec{x},t)] \Bigg\{ \frac{\partial v_\varphi(\vec{x},t)}{\partial t} + v_r(\vec{x},t) v_\varphi(\vec{x},t) \sin[\theta(\vec{x},t)]$$
$$+ v_\theta(\vec{x},t) v_\varphi(\vec{x},t) \cos[\theta(\vec{x},t)] + v_r(\vec{x},t) \frac{\partial v_\varphi(\vec{x},t)}{\partial r} + \frac{v_\theta(\vec{x},t)}{r} \frac{\partial v_\varphi(\vec{x},t)}{\partial \theta}$$
$$+ \frac{v_\varphi(\vec{x},t)}{r \sin\theta} \left[v_r(\vec{x},t) \sin\theta + v_\theta(\vec{x},t) \cos\theta + \frac{\partial v_\varphi(\vec{x},t)}{\partial \varphi} \right] \Bigg\}$$
$$+ \int d^3x\, \rho(\vec{x},t) r^2(\vec{x},t) \cos[\theta(\vec{x},t)] v_\theta(\vec{x},t) v_\varphi(\vec{x},t)$$
$$+ 2 \int d^3x\, \rho(\vec{x},t) r(\vec{x},t) v_r(\vec{x},t) \sin[\theta(\vec{x},t)] v_\varphi(\vec{x},t) = \sum_{i=1}^{2} F_\varphi^i(\vec{x},t). \tag{3.4.126}$$

So that now we have (3.4.117), (3.4.118), and (3.4.119) written in terms of "material derivative" which accounts for the time rate of change of a unit fluid volume at fixed position and spatial variation in fluid velocity at a fixed time. The formalism until now has dealt with a single continuous collection of streamlines scattering with a finite area of continuous mass distributions at one instant. This is the situation encountered in Figure (6). If we call that a single event, then to consider m such events we introduce the concept of "water beam", which is a continuous collection of streamlines moving all at the same time and scattering with the ground all at once. Therefore, Figure (6) is a single water beam. Although the water beam model resolves the problem of including many events of streamline-particle scattering, it brings up the problem of interactions between the water beams. The interactions between the beams is nonlinear which makes the application of the principle of superposition impossible. Particle-particle interactions are an example of interactions which are generated by the collisions between the particles that fly apart from different areas that different beams strike. Since different beams strike their interaction cross section at different times, a particle that flies off one area (due to beam ground interaction at one time) could collide with a streamline in a beam that would strike the ground at a later time. This brings in the particle-streamline interaction into our formalism. At this point we apply our Lagrangian density formalism to the active forces in phase II. Therefore, the problem of interactions due to the interference of water beams is resolved by finding an interaction Lagrangian density. Note the power of the Lagrangian density formalism of hydrodynamic forces we developed as it is already being applied in our hydrodynamic theory of dust transport. Before finding the interaction Lagrangian density, let us see what equations (3.4.124), (3.4.125), and (3.4.126) look like after considering m beams (or events).

Figure (7) below demonstrates a single beam representing a single event of dust scattering.

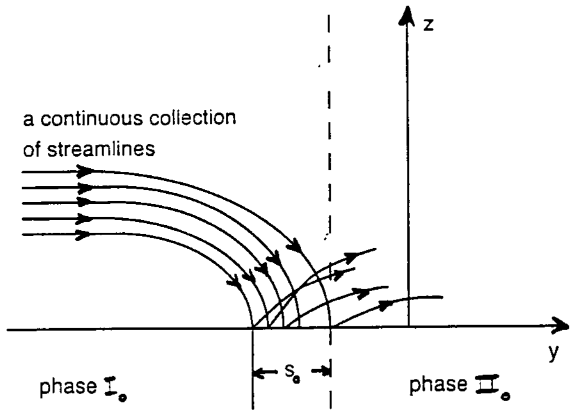

Figure 7: Showing a single water beam.

In order to visualize the interactions due to water beam interference, we show a figure with two beams each similar to the one shown in Figure (7). See Figure (8) below.

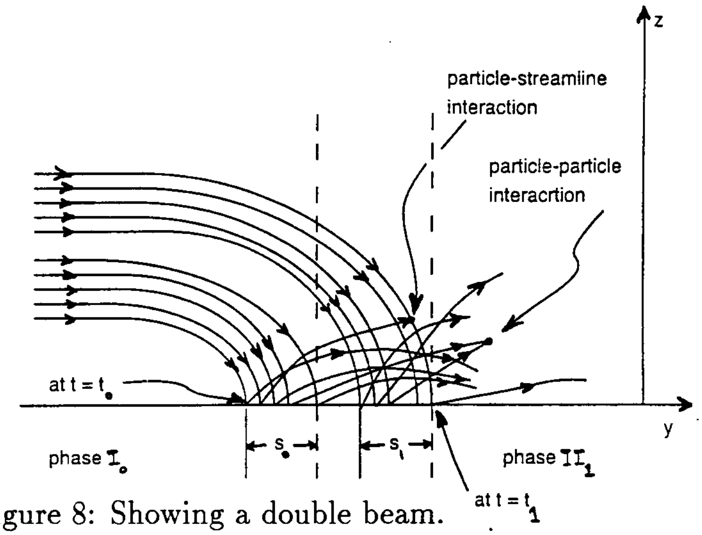

Figure 8: Showing a double beam.

Note the type of interactions generated as a result of considering more than one water beam. For m such water beams the picture would be m overlaps of Figure (7). For the kth water beam, equations (3.4.114) and (3.4.115) in spherical coordinates will become, respectively,

$$\left(F_r^{1^{(k)}}(\vec{x},t), F_\theta^{1^{(k)}}(\vec{x},t), F_\varphi^{1^{(k)}}(\vec{x},t)\right) \equiv$$

$$\int_{x_{k_i}}^{x_{k_f}} dx_k \int_{y_{k_i}}^{y_{k_f}} dy_k \Big(\cos\theta(x_k, y_k, t), -r(x_k, y_k, t)\sin\theta(x_k, y_k, t), 0\Big)$$
$$\times \int_{-\infty}^{\infty} d\tau\, \delta(\tau - t_k)$$
$$\times \Big\{[-\eta(\vec{\nabla} v_n) + p\hat{n} + \rho\vec{v}(\vec{v}\cdot\hat{n})]\cdot\hat{e}_3\Big\}(x_k, y_k), \tag{3.4.127}$$

and

$$\Big(F_r^{2(k)}(\vec{x},t), F_\theta^{2(k)}(\vec{x},t), F_\varphi^{2(k)}(\vec{x},t)\Big) =$$
$$\int_S \int_0^{z_0} dz_k \int_{-\infty}^{\infty} dx_k \Big(\varpi_r\big(r(x_k, z_k, t), \theta(x_k, z_k, t), \varphi(x_k, z_k, t)\big),$$
$$\varpi_\theta\big(r(x_k, z_k, t), \theta(x_k, z_k, t), \varphi(x_k, z_k, t)\big), \varpi_\varphi\big(r(x_k, z_k, t), \theta(x_k, z_k, t), \varphi(x_k, z_k, t)\big)\Big)$$
$$\times \int_t^{\infty} d\tau\, da(x_k, z_k) \int_{-\pi/2}^{\pi/2} d\vartheta(x_k, z_k) v^{inc}(x_k, z_k)\cos[\vartheta(x_k, z_k)]$$
$$\times \left[\Big|[\vec{\Pi}^c\cdot\hat{y}'](x_k, z_k)\Big| + \alpha\Big|[\vec{\Pi}^{c'}\cdot\hat{y}'](x_k, z_k)\Big|\right] \tag{3.4.128}$$

with $k = 0, 1, 2, \ldots, m$.

Equation (3.4.127) is then the force delivered to the continuous distribution of particles on area S_k of the ground by the continuous collection of streamlines scattering with the area S_k. On the other hand, (3.4.128) is the active force on the continuous particle distribution scattered by the kth water beam and diffused into phase II_k, where II_k refers to phase II for the kth particle. We will later find an interaction Lagrangian density, but for now assume that there exists an interaction Lagrangian density such that the total motion of the water beams can be described by superimposing the beams. Any non-additive effect can be included in the interaction Lagrangian density. We write equation (3.4.124) as

$$\sum_{k=0}^{m} \int d^3x_k\, \rho(\vec{x}_k, t)\Big\{\frac{\partial v_r(\vec{x}_k, t)}{\partial t} - [v_\theta(\vec{x}_k, t)]^2 - [v_\varphi(\vec{x}_k, t)]^2 \sin[\theta(\vec{x}_k, t)]$$

$$+ v_r(\vec{x}_k,t)\frac{\partial v_r(\vec{x}_k,t)}{\partial r} + \frac{v_\theta(\vec{x}_k,t)}{r}\left[\frac{\partial v_r(\vec{x}_k,t)}{\partial \theta} - v_\theta(\vec{x}_k,t)\right]$$
$$+ \frac{v_\varphi(\vec{x}_k,t)}{r\sin\theta}\left[\frac{\partial v_r(\vec{x}_k,t)}{\partial \varphi} - v_\varphi(\vec{x}_k,t)\sin[\theta(\vec{x}_k,t)]\right]$$
$$+ g\cos[\theta(\vec{x}_k,t)] - r(\vec{x}_k,t)[v_\theta(\vec{x}_k,t)]^2 - r(\vec{x}_k,t)\sin[\theta(\vec{x}_k,t)][v_\varphi(\vec{x}_k,t)]^2\Bigg\}$$
$$= \sum_{i=1}^{2}\sum_{k=0}^{m} F_r^{i(k)}(\vec{x},t). \qquad (3.4.129)$$

We now take equation (3.4.129) to a continuous limit, that is, consider $m \gg 1$, so equation (3.4.129) becomes

$$\int_{-\infty}^{\infty} d\zeta \int_{V(\zeta)} d^3x(\zeta)\rho(\vec{x}(\zeta),t)$$
$$\times\Bigg\{\frac{\partial v_r(\vec{x}(\zeta),t)}{\partial t} - [v_\theta(\vec{x}(\zeta),t)]^2 - [v_\varphi(\vec{x}(\zeta),t)]^2\sin[\theta(\vec{x}(\zeta),t)]$$
$$+ v_r(\vec{x}(\zeta),t)\frac{\partial v_r(\vec{x}(\zeta),t)}{\partial r} + \frac{v_\theta(\vec{x}(\zeta),t)}{r}\left[\frac{\partial v_r(\vec{x}(\zeta),t)}{\partial \theta} - v_\theta(\vec{x}(\zeta),t)\right]$$
$$+ \frac{v_\varphi(\vec{x}(\zeta),t)}{r\sin\theta}\left[\frac{\partial v_r(\vec{x}(\zeta),t)}{\partial \varphi} - v_\varphi(\vec{x}(\zeta),t)\sin[\theta(\vec{x}(\zeta),t)]\right] + g\cos[\theta(\vec{x}(\zeta),t)]$$
$$- r(\vec{x}(\zeta),t)[v_\theta(\vec{x}(\zeta),t)]^2 - r(\vec{x}(\zeta),t)\sin[\theta(\vec{x}(\zeta),t)][v_\varphi(\vec{x}(\zeta),t)]^2\Bigg\}$$
$$= \sum_{i=1}^{2}\int_{-\infty}^{\infty} d\zeta\, F_r^{i(\zeta)}(\vec{x},t), \qquad (3.4.130)$$

with ζ being the continuous variable replacing the discrete index k. Likewise equations (3.4.125) and (3.4.126) will become

$$\int_{-\infty}^{\infty} d\zeta \int_{V(\zeta)} d^3x(\zeta)\rho(\vec{x}(\zeta),t)\Bigg\{[r(\vec{x}(\zeta),t)]^2\frac{\partial v_\theta(\vec{x}(\zeta),t)}{\partial t} + [r(\vec{x}(\zeta),t)]^2 v_r(\vec{x}(\zeta),t)v_\theta(\vec{x}(\zeta),t)$$
$$+ [r(\vec{x}(\zeta),t)]^2 v_r(\vec{x}(\zeta),t)\frac{\partial v_\theta(\vec{x}(\zeta),t)}{\partial r} + 3r(\vec{x}(\zeta),t)v_\theta(\vec{x}(\zeta),t)v_r(\vec{x}(\zeta),t)$$
$$+ r(\vec{x}(\zeta),t)v_\theta(\vec{x}(\zeta),t)\ \frac{\partial v_\theta(\vec{x}(\zeta),t)}{\partial \theta}$$
$$+ r(\vec{x}(\zeta),t)\frac{v_\varphi(\vec{x}(\zeta),t)}{\sin\theta}\frac{\partial v_\theta(\vec{x}(\zeta),t)}{\partial \varphi} + r(\vec{x}(\zeta),t)[v_\varphi(\vec{x}(\zeta),t)]^2(\cos\theta - \cot\theta)$$

$$- gr(\vec{x}(\zeta),t)\sin\theta(\vec{x}(\zeta),t)\Big\} = \sum_{i=1}^{2}\int_{-\infty}^{\infty} d\zeta\, F_{\theta}^{i(\zeta)}(\vec{x},t) \tag{3.4.131}$$

and

$$\begin{aligned}
&\int_{-\infty}^{\infty} d\zeta \int_{V(\zeta)} d^3x(\zeta)\rho(\vec{x}(\zeta),t)[r(\vec{x}(\zeta),t)]^2\sin[\theta(\vec{x}(\zeta),t)]\Big\{\frac{\partial v_\varphi(\vec{x}(\zeta),t)}{\partial t}\\
&\quad + v_r(\vec{x}(\zeta),t)v_\varphi(\vec{x}(\zeta),t)\sin[\theta(\vec{x}(\zeta),t)]\\
&\quad + v_\theta(\vec{x}(\zeta),t)v_\varphi(\vec{x}(\zeta),t)\cos[\theta(\vec{x}(\zeta),t)] + v_r(\vec{x}(\zeta),t)\frac{\partial v_\varphi(\vec{x}(\zeta),t)}{\partial r}\\
&\quad + \frac{v_\theta(\vec{x}(\zeta),t)}{r}\frac{\partial v_\varphi(\vec{x}(\zeta),t)}{\partial\theta}\\
&\quad + \frac{v_\varphi(\vec{x}(\zeta),t)}{r\sin\theta}\left[v_r(\vec{x}(\zeta),t)\sin\theta + v_\theta(\vec{x}(\zeta),t)\cos\theta + \frac{\partial v_\varphi(\vec{x}(\zeta),t)}{\partial\varphi}\right]\Big\}\\
&\quad + \int_{-\infty}^{\infty} d\zeta \int_{V(\zeta)} d^3x(\zeta)\rho(\vec{x}(\zeta),t)[r(\vec{x}(\zeta),t)]^2\cos[\theta(\vec{x}(\zeta),t)]v_\theta(\vec{x}(\zeta),t)v_\varphi(\vec{x}(\zeta),t)\\
&\quad + 2\int_{-\infty}^{\infty} d\zeta \int_{V(\zeta)} d^3x(\zeta)\rho(\vec{x}(\zeta),t)r(\vec{x}(\zeta),t)v_r(\vec{x}(\zeta),t)\sin[\theta(\vec{x}(\zeta),t)]v_\varphi(\vec{x}(\zeta),t)\\
&= \sum_{i=1}^{2}\int_{-\infty}^{\infty} d\zeta\, F_{\varphi}^{i(\zeta)}(\vec{x},t),
\end{aligned} \tag{3.4.132}$$

where

$$\begin{aligned}
&\Big(F_r^{1(\zeta)}(\vec{x},t), F_\theta^{1(\zeta)}(\vec{x},t), F_\varphi^{1(\zeta)}(\vec{x},t)\Big) \equiv\\
&\quad \int_{x_i(\zeta)}^{x_f(\zeta)} dx(\zeta)\int_{y_i(\zeta)}^{y_f(\zeta)} dy(\zeta)\\
&\qquad \times\Big(\cos\theta\big(x(\zeta),y(\zeta),t\big), -r\big(x(\zeta),y(\zeta),t\big)\sin\theta\big(x(\zeta),y(\zeta),t\big), 0\Big)\\
&\qquad \times\int_{-\infty}^{\infty} d\tau\,\delta(\tau - t(\zeta))\\
&\qquad \times\Big\{[-\eta(\vec{\nabla}v_n) + p\hat{n} + \rho\vec{v}(\vec{v}\cdot\hat{n})]\cdot\hat{e}_3\Big\}(x(\zeta),y(\zeta))
\end{aligned} \tag{3.4.133}$$

and

$$\Big(F_r^{2(\zeta)}(\vec{x},t), F_\theta^{2(\zeta)}(\vec{x},t), F_\varphi^{2(\zeta)}(\vec{x},t)\Big) \equiv$$

$$\int_S \int_0^{z_0} dz(\zeta) \int_{-\infty}^{\infty} dx(\zeta) \Big(\varpi_r \Big(r\big(x(\zeta), z(\zeta), t\big), \theta\big(x(\zeta), z(\zeta), t\big), \varphi\big(x(\zeta), z(\zeta), t\big)\Big),$$
$$\varpi_\theta \Big(r\big(x(\zeta), z(\zeta), t\big), \theta\big(x(\zeta), z(\zeta), t\big), \varphi\big(x(\zeta), z(\zeta), t\big)\Big),$$
$$\varpi_\varphi \Big(r\big(x(\zeta), z(\zeta), t\big), \theta\big(x(\zeta), z(\zeta), t\big), \varphi\big(x(\zeta), z(\zeta), t\big)\Big)\Big)$$
$$\times \int_{t(\zeta)}^{\infty} d\tau da(x(\zeta), z(\zeta)) \int_{-\pi/2}^{\pi/2} d\vartheta(\vec{x}(\zeta), z(\zeta)) v^{inc}(x(\zeta), z(\zeta)) \cos[\vartheta(x(\zeta), z(\zeta))]$$
$$\times \Big[\Big|[\vec{\Pi}^c \cdot \hat{y}'](x(\zeta), z(\zeta))\Big| + \alpha \Big|[\vec{\Pi}^{c'} \cdot \hat{y}'](x(\zeta), z(\zeta))\Big| \Big]. \tag{3.4.134}$$

Equations (3.4.130), (3.4.131), and (3.4.132) are, respectively, the radial the polar and the azimuthal equations of motion for a continuous collection of water beams. We are not surprised that they have a complicated form. This is because the situation we were trying to describe was very complicated in the first place. Note the changes in (3.4.127) and (3.4.128) shown in (3.4.133) and (3.4.134). Let us now take the final step of chapter 3, which is determining an interaction Lagrangian density.

As simple as it sounds, determining an interaction Lagrangian density, $\mathscr{L}_{\mathrm{I}}$, turns out to be a complicated and a hard thing to do. Since the vector potential $\vec{A}$ pertains to forces due to streamlines in phases II_k, we cannot use it for the type of interactions generated by the multi-beam picture. So, we attempt to define a new velocity potential. We do that next in what we call the interactions of interferences.

3.5 Interactions of interferences

The theoretical development below, that led to the inclusion of interactions due to water beam interferences, is one of the most fascinating mathematical models we achieved. Not only were we able to use the continuity equation to describe scattered dust particles but also we rewrote the continuity equation, by proposing a particle flux equation with a nonlinear concentration term, in a form such that it possesses an amazing resemblance to the Schroedinger equation which immediately enabled us to apply the propagator theory. The nonlinear term of our Schroedinger like diffusion equation is beautifully interpreted as energy, which actually allowed the physical interpretation of the equation and the solution to it.

Since our equations of motion describe the propagation of dust particles in phases II_k, we can view this propagation as a diffusion process. That is, the particles diffusing into phases II_k. Now water is the media into which they diffuse. If the particles are fine dust or if the motion is violent enough, then the trajectories of the particles are similar in pattern to the streamline trajectories. Next, we define Φ , a field potential for the particles that resembles their concentration, with $\vec{v_p} = -\vec{\nabla}\Phi$ being the velocity of a particle. The minus sign is taken because the particles diffuse from a high concentration to a low concentration. Note that dust particles are being carried by streamlines and that $\Phi \neq \phi$, where ϕ is our previously defined scalar potential in $\vec{v} = -\vec{\nabla}\phi + \vec{\nabla} \times \vec{A}$. Since we are considering fine dust, we propose the following form of particle flux density:

$$\vec{q} = -D\vec{\nabla}\Phi + \kappa\vec{v}(\vec{x},t)\,(\Phi)^2\,, \tag{3.5.135}$$

where D and κ are constants, $\vec{q}$ is the flux of particles and $\vec{v}(\vec{x},t) = -\vec{\nabla}\phi(\vec{x},t) + \vec{\nabla} \times \vec{A}(\vec{x},t)$. The term Φ^2 reflects the fact that we are dealing with fine dust (highly diffusive) and their high rate of diffusion into the fluid. We express the high dispersion rate of particles into fluid as the $\vec{q}$'s direct proportionality to the nonlinear term Φ^2. We realize the fact that our constitution of the form of $\vec{q}$ in (3.5.135) is a new step that we proposed and gave a physical interpretation of. The step we proposed is supported by M. C. Bustos, F. Conchan, and W. L.

Wendland [5] as they bring up a general flux density function in their Kynch model of continuous sedimentation section. The flux density function, they mention, is a constitutive equation, and that supports our choice to construct a specific new particle flux density as in (3.5.135). Define $\dot{\Phi} \equiv \partial\Phi/\partial t$. From R. Guenther and J. Lee [15] the continuity equation for Φ is

$$\frac{\partial \Phi(\vec{x},t)}{\partial t} + \vec{\nabla} \cdot \vec{q}(\vec{x},t) = 0,$$

or

$$\dot{\Phi}(\vec{x},t) - D\nabla^2\Phi(\vec{x},t) + \kappa\vec{\nabla} \cdot \vec{v}(\vec{x},t)\,[\Phi(\vec{x},t)]^2 + 2\kappa\vec{v}(\vec{x},t) \cdot \vec{\nabla}\Phi(\vec{x},t)\Phi(\vec{x},t) = 0.$$

Note that $\vec{\nabla} \cdot \vec{v}_p \neq 0$, which means that an initial distribution of particles can diffuse from all sides of the enclosing surface. Remembering our incompressibility condition, we set $\vec{\nabla} \cdot \vec{v} = 0$ to have

$$\dot{\Phi}(\vec{x},t) - D\nabla^2\Phi(\vec{x},t) + 2\kappa\vec{v}(\vec{x},t) \cdot \vec{\nabla}\Phi(\vec{x},t)\Phi(\vec{x},t) = 0.$$

With $\vec{v_p} = -\vec{\nabla}\Phi$ the above equation can be written as

$$\dot{\Phi}(\vec{x},t) - D\nabla^2\Phi(\vec{x},t) = \kappa 2\vec{v} \cdot \vec{v}_p\Phi(\vec{x},t). \tag{3.5.136}$$

But $2\vec{v} \cdot \vec{v}_p = v_T^2 - v^2 - v_p^2$, where $\vec{v}_T \equiv \vec{v} + \vec{v}_p$. Therefore,

$$\rho\vec{v} \cdot \vec{v}_p = \frac{1}{2}\rho v_T^2 - \frac{1}{2}\rho v^2 - \frac{1}{2}\rho v_p^2. \tag{3.5.137}$$

The right hand side of (3.5.137) shows that the left hand side of (3.5.137) is the energy of the particle-streamline system minus the energy of both the particle and the streamline. In other words, there is an excess energy that is generated by the association of the particle with the streamline's trajectory. That energy accounts for the "glueness" of the particle to the streamline's path. It is that excess energy that gets transferred or absorbed during a collision with another particle-streamline system. So, while the right hand side of (3.5.137) tells us we have an energy term, the left hand side of (3.5.137) tells us that although the particles follow the streamline's velocity (momentum) path they are still not the same thing; the velocity of the two differ by the angle between them. Now if $|\vec{v}| = |\vec{v}_p|$ then, $\vec{v}\cdot\vec{v}_p = |\vec{v}_p|^2\cos\theta$ and if the particles are fine dust (point particles), i.e., we rule out angular velocity due to spin or rotation of the particle, then we have $-\epsilon\pi/2 \leq \theta \leq \epsilon\pi/2$, where $\epsilon \ll 1$. Then $|\theta| \ll 1$ which implies $\cos\theta \approx 1$. Consequently $\vec{v}\cdot\vec{v}_p = |\vec{v}_p|^2\cos\theta \approx |\vec{v}_p|^2$. Note that the excess energy term is similar in it's mathematical form to the energy generated due to the presence of a dipole in an external field. Therefore, equation (3.5.136) with the equality in (3.5.137) can be written as

$$\left(\frac{\partial}{\partial t} - D\nabla^2\right)\Phi(\vec{x},t) = V(\vec{x},t)\Phi(\vec{x},t), \tag{3.5.138}$$

where $V(\vec{x},t) \equiv (2\kappa/\rho)(\Delta KE)$, and where $\Delta KE \equiv 1/2\rho v_T^2 - 1/2\rho v^2 - 1/2\rho v_p^2$. Now that we identified $V(\vec{x},t)$ with energy we can apply the propagator theory. Note that proposing (3.5.135) led us successfully to (3.5.138) which have the appropriate physical interpretation for the propagator theory application. An experimental fact that supports our theoretical proposal of the particle flux's nonlinear dependence on concentration is given by G. J. Kynch [18]. In his paper, Kynch mentions that the particle's flux direct proportionality to the square of the concentration is in agreement with experimental curves he analyzed, however, he recedes to make any further conclusions about that fact as he considers it unwise to do so. Here, the significance of our theoretical result is emphasized by that experimental fact as we demonstrate the success of the application of the propagator theory by proposing a direct proportionality between particle flux and Φ^2. Before applying the propagator theory we summarize the development given in Bjorken and Drell [3] to get an n interaction propagator. The solution to equation (3.5.138) can be written in the form

$$\Phi(\vec{x}', t') = \psi(\vec{x}', t') + \Delta\Phi(\vec{x}', t'). \tag{3.5.139}$$

Here $\Phi(\vec{x}', t')$ is the solution at the point $(\vec{x}', t')$, $\psi(\vec{x}', t')$ is the free space solution to the diffusion equation at the point $(\vec{x}', t')$, and $\Delta\Phi(\vec{x}', t')$ is the change in the solution at $(\vec{x}', t')$ due to the presence of an interaction. The quantity $\Delta\Phi(\vec{x}', t')$ can be written as

$$\Delta\Phi(\vec{x}', t') = \int d^3x_1 G_0(\vec{x}', t'; \vec{x_1}, t_1)\Delta\Phi(\vec{x_1}, t_1). \tag{3.5.140}$$

$G_0(\vec{x}', t'; \vec{x_1}, t_1)$ is the propagator taking the solution from the point $(\vec{x_1}, t_1)$ to the point $(\vec{x}', t')$ without any interaction. Also,

$$\Delta\Phi(\vec{x_1}, t_1) = V(\vec{x_1}, t_1)\psi(\vec{x_1}, t_1)\Delta t_1 \tag{3.5.141}$$

and $V(\vec{x_1}, t_1)$ is the energy change at $(\vec{x_1}, t_1)$ per mass density. Using equation (3.5.140) we can write (3.5.139) as

$$\Phi(\vec{x}', t') = \psi(\vec{x}', t') + \int d^3x_1 G_0(\vec{x}', t'; \vec{x_1}, t_1)\Delta\Phi(\vec{x_1}, t_1).$$

From equation (3.5.141) this can be written as

$$\Phi(\vec{x}',t') = \psi(\vec{x}',t') + \int d^3x_1 \Delta t_1 G_0(\vec{x}',t';\vec{x_1},t_1)V(\vec{x_1},t_1)\psi(\vec{x_1},t_1).$$

By comparing this result with equation (3.5.140), we can write $\psi(\vec{x_1},t_1)$ as

$$\psi(\vec{x_1},t_1) = \int d^3x G_0(\vec{x_1},t_1;\vec{x},t)\psi(\vec{x},t);$$

therefore,

$$\Phi(\vec{x}',t') = \psi(\vec{x}',t') + \int d^3x \int d^3x_1 \Delta t_1 G_0(\vec{x}',t';\vec{x_1},t_1)V(\vec{x_1},t_1)G_0(\vec{x_1},t_1;\vec{x},t)\psi(\vec{x},t).$$

Similarly

$$\psi(\vec{x}',t') = \int d^3x G_0(\vec{x}',t';\vec{x},t)\psi(\vec{x},t),$$

so that

$$\begin{aligned}\Phi(\vec{x}',t') &= \int d^3x G_0(\vec{x}',t';\vec{x},t)\psi(\vec{x},t) \\ &\quad + \int d^3x \int d^3x_1\, \Delta t_1 G_0(\vec{x}',t';\vec{x_1},t_1)V(\vec{x_1},t_1)G_0(\vec{x_1},t_1;\vec{x},t)\psi(\vec{x},t)\end{aligned}$$

or

$$\begin{aligned}\Phi(\vec{x}',t') &= \int d^3x \Big[G_0(\vec{x}',t';\vec{x},t) \\ &\quad + \int d^3x_1 \Delta t_1 G_0(\vec{x}',t';\vec{x_1},t_1)V(\vec{x_1},t_1)G_0(\vec{x_1},t_1;\vec{x},t)\Big]\psi(\vec{x},t),\end{aligned}$$

which in turn can be written as

$$\Phi(\vec{x}',t') = \int d^3x G(\vec{x}',t';\vec{x},t)\psi(\vec{x},t), \tag{3.5.142}$$

where

$$G(\vec{x}',t';\vec{x},t) = G_0(\vec{x}',t';\vec{x},t) + \int d^3x_1 \Delta t_1 G_0(\vec{x}',t';\vec{x_1},t_1)V(\vec{x_1},t_1)G_0(\vec{x_1},t_1;\vec{x},t) \tag{3.5.143}$$

is the propagator taking the solution from $(\vec{x},t)$ to $(\vec{x}',t')$. The following picture depicts the kind of physics happening in equation (3.5.143).

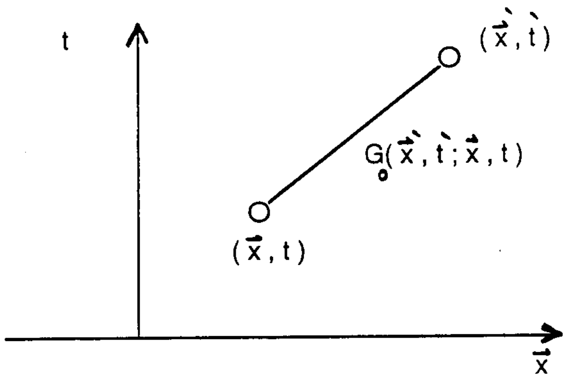

Figure 9: The first term of the right hand side of (3.5.143).

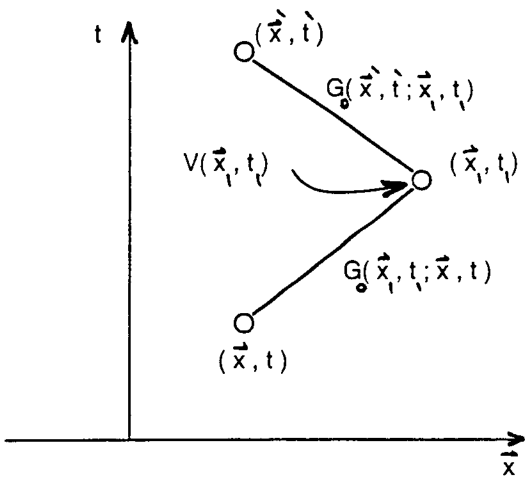

Figure 10: The second term of the right hand side of (3.5.143).

Since our boundary condition on G_0 is propagation forward in time (i.e. a retarded propagator), we set $G_0(\vec{x}',t';\vec{x},t) = 0$ for $t' < t$. Consider n interactions similar to the one in Figure (10). Equation (3.5.143) becomes in that case

$$\begin{aligned} G(\vec{x}',t';\vec{x},t) &= G_0(\vec{x}',t';\vec{x},t) \\ &+\sum_i \int d^3x_i \Delta t_i G_0(\vec{x}',t';\vec{x}_i,t_i)V(\vec{x}_i,t_i)G_0(\vec{x}_i,t_i;\vec{x},t) \\ &+ \sum_{\substack{i>j \\ (t_i>t_j)}} \int d^3x_i\Delta t_i d^3x_j \Delta t_j G_0(\vec{x}',t';\vec{x}_i,t_i)V(\vec{x}_i,t_i) \\ &\times G_0(\vec{x}_i,t_i;\vec{x}_j,t_j)V(\vec{x}_j,t_j)G_0(\vec{x}_j,t_j;\vec{x},t) + \cdots . \qquad (3.5.144) \end{aligned}$$

Since we have n interactions in equation (3.5.144), we can choose them to be either particle-particle or particle-streamline. Since we want interactions due to interferences of beams the following model demonstrated in the figure below should be satisfactory.

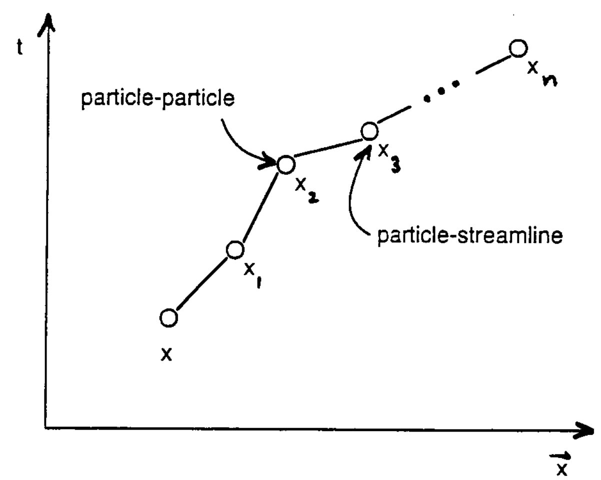

Figure 11: Some of the possible interactions are shown in this figure.

Figure (11) is the result of our interactions due to the mathematical description of the interferences. This situation is reflected pictorially in Figure (8). Therefore, our formalism neatly describes the interactions generated by the multi-beam picture and introduces a new velocity scalar potential field, which means we can write out an interaction Lagrangian density $\mathscr{L}_I$. By way of example, we assume

$\mathscr{L}_{\mathrm{I}}$ to be of the form

$$\mathscr{L}_{\mathrm{I}} = \frac{1}{2}(\dot{\Phi})^2. \tag{3.5.145}$$

To be able to incorporate (3.5.145) into (3.4.134) we define a five term, canonical momentum density. Equation (3.3.86) would then become

$$\vec{\Pi}_k^c = \sum_{j=0}^{4} \frac{\partial \mathscr{L}}{\partial \dot{\eta}_{jk}} \hat{e}_j \tag{3.5.146}$$

where $\mathscr{L} = \mathscr{L}_0 + \mathscr{L}_{\mathrm{I}}$ and where

$$\eta_{jk} = \begin{cases} \phi_k, & \text{if } j = 0, \\ A_{1k}, & \text{if } j = 1, \\ A_{2k}, & \text{if } j = 2, \\ A_{3k}, & \text{if } j = 3, \\ \Phi_k, & \text{if } j = 4. \end{cases}$$

Also $\hat{e_4} = \sum b_j \hat{e_j}$ is a unit vector in the direction of propagation of the canonical momentum density of Φ_k with b_j being the direction cosine of that vector. Expanding (3.5.146) we get

$$\begin{aligned} \vec{\Pi}_k^c &= \sum_{j=0}^{4} \frac{\partial \mathscr{L}}{\partial \dot{\eta}_{jk}} \hat{e}_j = \frac{\partial \mathscr{L}}{\partial \dot{\eta}_{0k}} \hat{e}_0 + \frac{\partial \mathscr{L}}{\partial \dot{\eta}_{1k}} \hat{e}_1 + \frac{\partial \mathscr{L}}{\partial \dot{\eta}_{2k}} \hat{e}_2 + \frac{\partial \mathscr{L}}{\partial \dot{\eta}_{3k}} \hat{e}_3 + \frac{\partial \mathscr{L}}{\partial \dot{\eta}_{4k}} \hat{e}_4 \\ &= \Pi_0^c \hat{e}_0 + \Pi_1^c \hat{e}_1 + \Pi_2^c \hat{e}_2 + \Pi_3^c \hat{e}_3 + \Pi_4^c \hat{e}_4. \end{aligned}$$

Equations (3.3.87) and (3.3.88) are, respectively,

$$|\vec{\Pi}_k^c \cdot \hat{y}'| = \left| \sum_{j=1}^{3} (a_j \frac{\partial \mathscr{L}}{\partial \dot{\phi}_k} + \frac{\partial \mathscr{L}}{\partial \dot{A}_{jk}} + b_j \frac{\partial \mathscr{L}}{\partial \dot{\Phi}_k})(\hat{e}_j \cdot \hat{y}') \right| \tag{3.5.147}$$

and

$$|\vec{\Pi}_k^{c'} \cdot \hat{y}'| = \left|\sum_{j=1}^{3}(a'_j \frac{\partial \mathscr{L}}{\partial \dot{\phi}'_k} + \frac{\partial \mathscr{L}}{\partial \dot{A}'_{jk}} + b'_j \frac{\partial \mathscr{L}}{\partial \dot{\Phi}'_k})(\hat{e}_j \cdot \hat{y}')\right|, \tag{3.5.148}$$

where the primes are used for reflected quantities. From (3.3.98) and (3.5.145) $\mathscr{L}$ can then be written as

$$\begin{aligned} \mathscr{L} &= \Bigg[\dot{\phi}_k + \frac{1}{2}(\vec{\nabla}\phi_k)^2 + gz + \frac{1}{2}\rho(\dot{A}_{jk})^2 - \frac{1}{2}\eta(\vec{\nabla}A_{jk})^2 \\ &\quad - \rho\sum_{n=1}^{3}\alpha_{jn}\dot{A}_{nk}A_{jk} - \lambda\sum_{n=1}^{3}(\hat{t}\cdot\vec{\nabla}\alpha_{jn})\dot{A}_{nk}A_{jk} + \frac{1}{2}(\dot{\Phi}_k)^2\Bigg], \end{aligned} \tag{3.5.149}$$

which means that (3.5.147) and (3.5.148) can be written as

$$\begin{aligned} |\vec{\Pi}_k^{c} \cdot \hat{y}'| &= \Bigg|\sum_{j=1}^{3}\Big\{\Big(a_j + \rho\dot{A}_{jk}(\vec{x},t) - \rho\alpha_{jj}A_{jk}(\vec{x},t) \\ &\quad - \lambda(\hat{t}\cdot\vec{\nabla}\alpha_{jj})A_{jk}(\vec{x},t) + b_j\dot{\Phi}_k(\vec{x},t)\Big)\Big\}(\hat{e}_j\cdot\hat{y}')\Bigg| \end{aligned} \tag{3.5.150}$$

and

$$\begin{aligned} |\vec{\Pi}_k^{c'} \cdot \hat{y}'| &= \Bigg|\sum_{j=1}^{3}\Big\{\Big(a'_j + \rho\dot{A}'_{jk}(\vec{x},t) - \rho\alpha_{jj}A'_{jk}(\vec{x},t) \\ &\quad - \lambda(\hat{t}\cdot\vec{\nabla}\alpha_{jj})A'_{jk}(\vec{x},t) + b'_j\dot{\Phi}'_k(\vec{x},t)\Big)\Big\}(\hat{e}_j\cdot\hat{y}')\Bigg|, \end{aligned} \tag{3.5.151}$$

respectively. The primes, as before, are used for reflected quantities. Observe that there is not much change in the structure of the equations. The effect of the inclusion of $\dot{\Phi}_k(\vec{x},t)$ in (3.3.98) is just the addition of a single term to the previously found equations without the $\dot{\Phi}_k(\vec{x},t)$.

Recapitulating our results and equations we have the following final statements as our theoretical modeling of the hydrodynamics of mass transport along water waves. The radial, the polar, and the azimuthal equations of motion of the scattered particles are given by (3.4.130), (3.4.131), and (3.4.132), respectively. The right hand side of the above equations are the hydrodynamic forces given by (3.4.133) and (3.4.134). The absolute values of the incident and the reflected components of the canonical momentum densities along $\hat{y}'$ in (3.4.134) are given by the continuous limit of (3.5.150) and (3.5.151), respectively. The Lagrangian density appearing in (3.5.147) and (3.5.148) is the one given in (3.5.149). Here the quantity ϕ_k in (3.5.149) is the solution to the equation (3.1.27) for the kth particle, A_{jk} are the solutions to (3.3.97) and Φ_k is given by (3.5.142) for the kth particle. The quantity $\Phi(\vec{x}(\zeta), t)$ would be the continuous form of (3.5.142) for the kth particle and the quantity $G(\vec{x}'(\zeta), t'; \vec{x}(\zeta), t)$ in $\Phi(\vec{x}(\zeta), t)$ would then be the continuous limit of the one given in (3.5.144). Therefore, we have a full, a descriptive, and a complete theory of the transport of mass particles by hydrodynamic forces.

4 Matter Forces of Water

4.1 Water surface tension

In this chapter we focus on the chemical nature of water rather than it's mechanical properties as a fluid. As we shall see in chapter 5 knowing the nature of the chemical bonds in water plays an important role in determining some dynamical variables. Let us for the moment view water as a discrete collection, rather than a bulk, of matter, that is, a discrete collection of molecules or water molecules. Next we ask what type of forces exist between these water molecules that makes a liquid drop of water exhibit a spherical shape. It turns out that the spherical shape is due to the existence of surface tension in water. By studying surface tension of water we are studying the chemical nature of the myriad collection of discrete water molecules. It turns out that there isn't much work available about water surface tension that gives a fundamental analysis of the forces responsible for surface tension. Therefore, we have developed our own theory and checked it's compatibility with what is little known about the phenomenon of surface tension. In our theory we start with Sears, Zemansky, and Young's [24] assertion that at a constant temperature, when a thin film of soapy water is stretched, the tension in the thin film does not increase appreciably from its previous or initial value. The reason for this is because we can treat the thin film as a layer with a definite, though very small, thickness. When the thin film is stretched, molecules on the surface are pulled apart and are replaced by the molecules that were beneath the molecules on the surface. The following figures demonstrate the idea:

Figure 12: Showing unbent thin film.

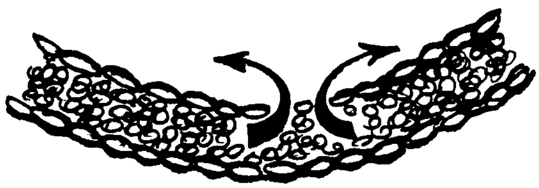

Figure 13: A bent thin film with arrows representing volume molecules moving from inside to outside layer.

We develop this line of reasoning based on the figure shown below. We apply it to a volume of water in a container which is open to the atmosphere. If we freeze the time and view a one dimensional cross-section of the wavy surface motion at that instant, it would have the following shape

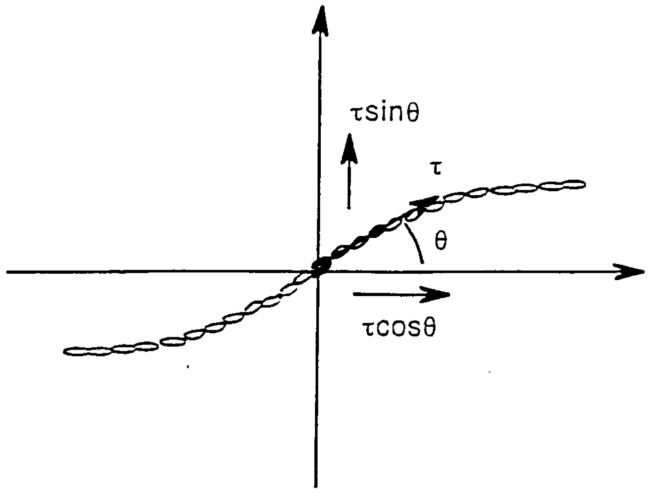

Figure 14: This figure shows the molecules of a wavy water. Note the dark molecule and the forces of tension shown above due to the wavy motion.

The balance of the forces applied to it is described as follows

$$-\tau \cos\theta = \text{Force due to adjacent molecule}$$

One question immediately arises and that is: what type of force does the adjacent molecule exert on the dark molecule?

We know from its chemical structure that a molecule of water has a permanent dipole, because the hydrogen atoms are electropositive and the oxygen atom is electronegative. This makes the interaction between the dark and the adjacent molecule a dipole-dipole one. From classical electrodynamics (see D. Jackson [16] p. 143) the potential between the two dipoles in electrostatic units is

$$U_{12} = \frac{\vec{\mu_1} \cdot \vec{\mu_2} - 3(\hat{n} \cdot \vec{\mu_1})(\hat{n} \cdot \vec{\mu_2})}{r^3}, \tag{4.1.1}$$

$$U_{12} = U_{1\to 2} = \text{potential energy of dipole 2 due to dipole 1}$$

$\hat{n}$ = unit vector in the direction joining the centers of the two dipoles

μ_1 = first dipole

μ_2 = second dipole

r = radial distance between the two dipoles.

A dipole can be represented by an arrow with its head pointing toward the positive charge cloud and its tail at the negative charge cloud (see Figure (15) below). Since we approximated a water molecule by a dipole, then the water molecule can also be represented by an arrow of the same type corresponding to its dipole. In the discussion below, we shall restrict ourselves to the case of a one dimensional surface bounding a two dimensional fluid.

Figure 15: A prolate and an equivalent arrow representing a dipole.

There are several possible arrangements which the dipoles can assume. Among those which are essentially distinct are the following arrangements which can arise.

$\rightarrow\rightarrow\rightarrow\rightarrow$ Arrangement resulting in attraction between dipoles.

$\uparrow\uparrow\uparrow\uparrow$ Arrangement resulting in repulsion between dipoles.

$\rightarrow\leftarrow\rightarrow\leftarrow$ Arrangement resulting in repulsion between dipoles.

$\uparrow\downarrow\uparrow\downarrow$ Arrangement resulting in attraction between dipoles.

The arrangement we chose for our model in Figure (15) is the least energetic one, which is the first arrangement. That is, the head to tail parallel dipole arrangement. Consequently, we claim that water surface tension is due to the arrangement being parallel as shown above.

The potential energy due to the parallel arrangement of two dipoles is by equation (4.1.1):

$$U_{12}(r) = -\frac{2\mu^2}{r^3},$$

where for the time being we set $\mu_1 = \mu_2 = \mu$ and the index is as described before in equation (4.1.1) which is potential energy of dipole 1 due to dipole 2 or potential energy of dipole 2 due to dipole 1. This interpretation is possible because of the symmetry of the potential energy equation with respect to the interchange of the two dipoles. The conservative force equation is then

$$F_{12} = -\frac{dU_{12}}{dr} = -\frac{6\mu^2}{r^4},$$

where F_{12} is the attractive force between dipoles 1 and 2.

From Figure (11) we have

$$\begin{aligned} -\tau\cos\theta &= \text{force due to dipole-dipole} \\ -\tau\cos\theta &= -\frac{6\mu\mu'}{r^4} \end{aligned}$$

where we substitute the mean dipole moment (see [26]) for the permanent dipole moment of the dark (kth) molecule. In other words, we are assuming all the permanent dipoles on the surface to be parallel and allow the kth dipole to rotate half space about the line of parallel permanent dipoles. The above equation can be written as

$$\tau = \frac{6}{\cos\theta}\frac{\mu\mu'}{r^4}. \tag{4.1.2}$$

From [16] and [31] the electric field due to a dipole in electrostatic units (esu) is $\vec{E}_D = [3(\vec{\mu}\cdot\hat{r})\hat{r} - \vec{\mu}]/r^3$. If $\vec{\mu} = \mu\hat{r}$, we then have $\vec{E}_D = 2\mu\hat{r}/r^3$, where $\hat{r}$ is a unit vector in the radial direction. Also from [26], pages 365–367, the mean dipole moment is $\mu' = \alpha E_0$ where α is the polarizability of a permanent dipole in an external field and E_0 is the applied electrical field. If the applied field is due to another dipole then we write the mean dipole moment as $\mu' = \alpha E_D = 2\alpha\mu/r^3$. Equation (4.1.2) would then become

$$\tau = \frac{6}{\cos\theta}\frac{\mu}{r^4}\frac{2\alpha\mu}{r^3} = \frac{12\alpha}{\cos\theta}\frac{\mu^2}{r^7} \tag{4.1.3}$$

in electrostatic units.

The above equation for τ is our first mathematical equation if the surface tension of water is generated as a result of bending. We note that it is a van der Waals type force with an attenuation factor to account for the bending or the departure of the considered surface from the equilibrium undeformed surface. Equation (4.1.3) is based on the nearest neighbor interaction. We now develop a model that is more sophisticated than the one given in equation (4.1.3). From [26], page 192, we find that the classical partition function for a one dimensional van der Waals fluid is:

$$Z = \left(\frac{e}{Nh}\right)^N \int \cdots \int \exp\left[-\beta H\right] dq_1 \cdots dp_N, \qquad (4.1.4)$$

where $\beta = \frac{1}{\kappa_B T}$, κ_B = Boltzmann constant, h = constant, T = temperature, and $H = \sum_{i=1}^{N} \frac{p_i^2}{2m} + \sum_{\substack{i,j \\ i \neq j}} V(r_{ij})$ is the Hamiltonian of the system and $V(r_{ij})$ is the van der Waals interaction energy (potential energy). The integration is over all values of the coordinates and momenta, i.e., $[-\infty, \infty]$. Z can be integrated with respect to the momentum and the generalized coordinates separately. This yields

$$\begin{aligned} Z &= \left(\frac{e}{Nh}\right)^N \int \cdots \int dp_1 \cdots dp_N \exp\left[-\beta \sum_{i=1}^{N} \frac{p_i^2}{2m}\right] \int \cdots \int dq_1 \cdots dq_N \exp\left[-\beta \sum_{i<j} V_{ij}\right] \\ &= \left[\frac{e}{Nh} \int_{-\infty}^{\infty} dp_i \exp\left[-\beta p_i^2/2m\right]\right]^N \int \cdots \int dq_1 \cdots dq_N \exp\left[-\beta \sum_{i<j} V_{ij}\right], \end{aligned}$$

where

$$\frac{1}{h} \int_{-\infty}^{\infty} dp_i \exp\left[-\beta p_i^2/2m\right] = \frac{(2\pi m \kappa_B T)^{1/2}}{h} \equiv \frac{1}{\lambda},$$

and λ from [22] is $\lambda \equiv (2\pi m \kappa_B T)^{1/2}/h$ and is called the thermal wavelength. The partition function is then

$$Z = \left(\frac{e}{N\lambda}\right)^N \int dq_1 \cdots \int dq_N \exp\left[-\beta \sum_{i<j} V(r_{ij})\right]. \qquad (4.1.5)$$

We now define a new function by dividing equation (4.1.5) by $(e/N\lambda)^N$ and call it the configuration function. Therefore

$$Q(T) = \left(\frac{N\lambda}{e}\right)^N Z(T) = \int dq_1 \cdots \int dq_N \exp\left[-\beta \sum_{i<j} V(r_{ij})\right] \qquad (4.1.6)$$

is our defined configuration function.

From Kittel and Kroemer [17], page 61, the internal energy due to interactions is

$$U(T) \quad = \quad \kappa_B T^2 \frac{\partial \ln Z(T)}{\partial T}, \tag{4.1.7}$$

and from Figure (15) we have

$$\begin{aligned} -\tau_k \cos\theta_k &= \text{Force due to molecular chain of water on the right hand side.} \\ \tau_k &= \text{Tension in the surface (1-dimensional) at the kth molecular site.} \end{aligned}$$

We will drop the k subscript for convenience and introduce a new internal energy function $U = U(r, T)$. The force applied on the kth molecule due to the molecular chain is then equal to $-\partial U(r,T)/\partial r$. From Newton's equation of motion and from the analysis above, we have

$$-\tau \cos\theta = -\frac{\partial U(r,T)}{\partial r}. \tag{4.1.8}$$

It is clear from equation (4.1.8) that the introduction of $U(r,T)$ was necessary. Since the resulting equation corresponding to (4.1.7) will have Z involved in it, we find it necessary to introduce a new partition function $Z = Z(r,T)$. To make Z a function of r we integrate the coordinate of the kth molecule to r. This makes the right hand side of (4.1.8) nonvanishing. From (4.1.8) we have

$$\tau \quad = \quad \frac{1}{\cos\theta}\frac{\partial U(r,T)}{\partial r},$$

or

$$\tau \quad = \quad \frac{1}{\cos\theta} k_B T^2 \frac{\partial^2 \ln Z(r,T)}{\partial r \partial T}. \tag{4.1.9}$$

From (4.1.6) we have

$$\ln Z(r,T) = \ln\left\{\left(\frac{e}{N\lambda}\right)^N Q(r,T)\right\} = N\ln\left(\frac{e}{N\lambda}\right) + \ln Q(r,T),$$

and because of (4.1.9) we take the partials $\partial^2 \ln Z/\partial r\partial T$. However, $\partial^2 \ln Z/\partial r\partial T = \partial^2 \ln Z/\partial T\partial r$, so that

$$\frac{\partial^2 \ln Z}{\partial r\partial T} = \frac{\partial^2 \ln Z}{\partial T\partial r} = \frac{\partial}{\partial T}\frac{\partial \ln Z}{\partial r} = \frac{\partial}{\partial T}\left\{\frac{\partial}{\partial r}\left[N\ln(\frac{e}{N\lambda}) + \ln Q(r,T)\right]\right\}.$$

Since $\lambda = \lambda(T)$ only, we have

$$\frac{\partial^2 \ln Z}{\partial r\partial T} = \frac{\partial}{\partial T}\frac{\partial \ln Q(r,T)}{\partial r} = \frac{\partial^2 \ln Q(r,T)}{\partial r\partial T}.$$

Therefore, we have the very important relation that

$$\frac{\partial^2 \ln Z}{\partial r\partial T} = \frac{\partial^2 \ln Q}{\partial r\partial T}, \tag{4.1.10}$$

which leads to the result

$$\tau = \frac{1}{\cos\theta}\frac{\partial U(r,T)}{\partial r} = \frac{k_B T^2}{\cos\theta}\frac{\partial^2 \ln Z(r,T)}{\partial r\partial T} = \frac{k_B T^2}{\cos\theta}\frac{\partial^2 \ln Q(r,T)}{\partial r\partial T}. \tag{4.1.11}$$

Note that

$$\frac{\partial^2 \ln Q(r,T)}{\partial r\partial T} = \frac{\partial}{\partial r}\frac{1}{Q}\frac{\partial Q}{\partial T} = -\frac{\partial Q}{\partial r}\frac{1}{Q^2}\frac{\partial Q}{\partial T} + \frac{1}{Q}\frac{\partial^2 Q}{\partial r\partial T}. \tag{4.1.12}$$

Let us focus on the kth molecule (the dark molecule) and make use of the fact that the van der Waals potential is decaying extremely rapidly. Then the integral over the kth coordinate can approximately be written as

$$\int dq_k \exp\left[-\beta \sum_{i\neq k} V_{ik}\right] = \int dq_k \exp\left[-\beta V(r_k)\right],$$

where the summation sign is dropped because of the rapidly decaying nature of the van der Waals potential and r_k is the radial coordinate of the kth molecule. We put the kth molecule at the origin of the coordinates and integrate over the interval $J = [r_0, r]$ where we treat r as a variable and take r_0 to be a scale related constant, the choice of which is explained later. Dropping the indices, we then have

$$\int dq_k \exp\left[-\beta V(r_k)\right] = \int_{r_0}^{r} dr \exp\left[-\beta V(r)\right]. \tag{4.1.13}$$

With these changes, (4.1.6) can be written as

$$\begin{aligned}
Q(r,T) &= \int dq_1 \ldots \int dq_{N-1} \int dq_k \exp\left[-\beta \sum_{i,j} V_{ij}\right] \\
&= \int dq_1 \ldots \int dq_{N-1} \int dq_k \exp\left[-\beta {\sum_{i,j}}' V_{ij}\right] \exp\left[-\beta \sum_{i\neq j} V_{ij}\right] \\
&= \int dq_1 \ldots \int dq_{N-1} \int dq_k \exp\left[-\beta {\sum_{i,j}}' V_{ij}\right] \exp\left[-\beta V(r_{ik})\right] \\
&= \int dq_1 \ldots \int dq_{N-1} \exp\left[-\beta {\sum_{i,j}}' V_{ij}\right] \int_{r_0}^{r} dr \exp\left[-\beta V(r)\right], \qquad (4.1.14)
\end{aligned}$$

where $\sum_{i,j}' V_{ij}$ is the symmetric summation taken over the rest of the pairs excluding the kth molecule. Our $Q(r,T)$ is now in a form we can use, so that

$$\begin{aligned}
\frac{\partial Q}{\partial r} &= \int dq_1 \ldots \int dq_{N-1} \exp\left[-\beta {\sum_{ij}}' V_{ij}\right] \int_{r_0}^{r} dr \left[-\beta \frac{dV(r)}{dr}\right] \exp\left[-\beta V(r)\right] \\
&\quad + \int dq_1 \ldots \int dq_{N-1} \exp\left[-\beta {\sum_{ij}}' V_{ij}\right] \exp\left[-\beta V(r)\right] \qquad (4.1.15)
\end{aligned}$$

is non-zero.

Set

$$\int dq_1 \ldots \int dq_{N-1} \exp\left[-\beta \sum_{ij}' V_{ij}\right] \exp\left[-\beta V(r)\right] \equiv \mathbb{I}. \tag{4.1.16}$$

We would like to note here that after we differentiate Q with respect to r, we let the upper limit of the integral tend to ∞. The integral (4.1.15) then takes the form

$$\begin{aligned}
\frac{\partial Q}{\partial r} &= \int dq_1 \ldots \int dq_{N-1} \exp\left[-\beta \sum_{ij}' V_{ij}\right] \int_{r_0}^{\infty} \left[-\beta \frac{dV}{dr}\right] \exp\left[-\beta V(r)\right] dr + \mathbb{I} \\
&= \int dq_1 \int dq_2 \ldots \int dq_N \left[-\beta \frac{dV}{dr}\right] \exp\left[-\beta \sum_{ij} V_{ij}\right] + \mathbb{I}. \qquad (4.1.17)
\end{aligned}$$

Next differentiate Q with respect to T.

$$\begin{aligned}
\frac{\partial Q}{\partial T} &= \int dq_1 \ldots \int dq_N \left[\frac{1}{k_B T^2} \sum_{ij} V_{ij}\right] \exp\left[-\beta \sum_{ij} V_{ij}\right] \\
&= \int dq_1 \ldots \int dq_{N-1} \exp\left[-\beta \sum_{ij}' V_{ij}\right] \int dq_k \left[\frac{1}{k_B T^2} \sum_{ij} V_{ij}\right] \exp\left[-\beta V(r_{ik})\right] \\
&= \int dq_1 \ldots \int dq_{N-1} \exp\left[-\beta \sum_{ij}' V_{ij}\right] \int_{r_0}^{r} dr \left[\frac{1}{k_B T^2} \sum_{ij} V_{ij}\right] \exp\left[-\beta V(r)\right].
\end{aligned}$$

Recall that r is the coordinate of the kth molecule. So that we have

$$\begin{aligned}
\frac{\partial^2 Q}{\partial r \partial T} &= \int dq_1 \ldots \int dq_{N-1} \exp\left[-\beta \sum_{ij}' V_{ij}\right] \int_{r_0}^{r} dr \frac{\partial}{\partial r} \left[\frac{1}{k_B T^2} \sum_{ij} V_{ij} \exp\left[-\beta V(r)\right]\right] \\
&\quad + \int dq_1 \ldots \int dq_{N-1} \exp\left[-\beta \sum_{ij}' V_{ij}\right] \frac{1}{k_B T^2} \sum_{ij} V_{ij} \exp\left[-\beta V(r)\right]. \qquad (4.1.18)
\end{aligned}$$

Let us take a look at the second term of (4.1.18)

$$\int dq_1 \ldots \int dq_{N-1} \exp\left[-\beta \sum_{ij}^{\prime} V_{ij}\right] \frac{1}{k_B T^2} \sum_{ij} V_{ij} \exp\left[-\beta V(r)\right].$$

Since we divide (4.1.18) by Q in (4.1.12) the above relation becomes

$$\begin{aligned}
& \frac{\beta}{T} \frac{\int dq_1 \ldots \int dq_{N-1} \left(\sum_{ij} V_{ij}\right) \exp\left[-\beta \sum_{ij}^{\prime} V_{ij}\right] \exp\left[-\beta V(r)\right]}{\int dq_1 \ldots \int dq_N \exp\left[-\beta \sum_{ij} V_{ij}\right]} \\
= \; & \frac{\beta}{T} \frac{\left[\int dq_1 \ldots \int dq_{N-1} \left(\sum_{ij}^{\prime} V_{ij}\right) \exp\left[-\beta \sum_{ij}^{\prime} V_{ij}\right]\right] \exp\left[-\beta V(r)\right]}{\int dq_1 \ldots \int dq_{N-1} \exp\left[-\beta \sum_{ij}^{\prime} V_{ij}\right] \int_{r_0}^{r} dr \exp\left[-\beta V(r)\right]} \\
& + \frac{\beta}{T} \frac{\left[\int dq_1 \ldots \int dq_{N-1} V(r) \exp\left[-\beta \sum_{ij}^{\prime} V_{ij}\right]\right] \exp\left[-\beta V(r)\right]}{\int dq_1 \ldots \int dq_{N-1} \exp\left[-\beta \sum_{ij}^{\prime} V_{ij}\right] \int_{r_0}^{r} dr \exp\left[-\beta V(r)\right]} \\
= \; & \frac{\beta}{T} \left[\left\langle \sum_{ij}^{\prime} V_{ij} \right\rangle^{\star} + \langle V(r) \rangle^{\star}\right] \frac{\exp\left[-\beta V(r)\right]}{\int_{r_0}^{r} dr \exp\left[-\beta V(r)\right]},
\end{aligned} \tag{4.1.19}$$

where from [22], page 106, the quantity $\frac{1}{Q} \exp\left[-\beta \sum_{ij} V_{ij}\right]$ is the probability that N particles are in the states specified by the function in the numerator. Accordingly, the average of a variable A is

$$\langle A \rangle = \frac{\int dq_1 \ldots \int dq_N A(x) \exp\left[-\beta \sum_{ij} V_{ij}\right]}{Q}.$$

The star implies that the average is taken over the coordinates of $N-1$ particles. However,

$$\begin{aligned}
\frac{\exp\left[-\beta V(r)\right]}{\int_{r_0}^{r} dr \exp\left[-\beta V(r)\right]} &= \frac{\exp\left[-\beta V(r)\right]}{\int_{r_0}^{r} dr \exp\left[-\beta V(r)\right]} \frac{\int dq_1 \ldots \int dq_{N-1} \exp\left[-\beta \sum_{ij}^{\prime} V_{ij}\right]}{\int dq_1 \ldots \int dq_{N-1} \exp\left[-\beta \sum_{ij}^{\prime} V_{ij}\right]} \\
&= \frac{\int dq_1 \ldots \int dq_{N-1} \exp\left[-\beta \sum_{ij}^{\prime} V_{ij}\right] \exp\left[-\beta V(r)\right]}{\int dq_1 \ldots \int dq_N \exp\left[-\beta \sum_{ij} V_{ij}\right]}.
\end{aligned}$$

By using equations (4.1.6) and (4.1.16), we can write the above relation as

$$\frac{\exp\left[-\beta V(r)\right]}{\int_{r_0}^{r} dr \exp\left[-\beta V(r)\right]} = \frac{\mathbb{I}}{Q}. \tag{4.1.20}$$

Now we go back to the first term in (4.1.18) and evaluate the derivative separately.

$$\begin{aligned}
&\frac{\partial}{\partial r}\left(\frac{1}{k_B T^2}\sum_{ij} V_{ij}\exp\left[-\beta V(r)\right]\right) \\
&\quad = \frac{1}{k_B T^2}\left[\frac{dV(r)}{dr}\exp\left[-\beta V(r)\right] + \sum_{ij} V_{ij}\left(-\beta\frac{dV}{dr}\right)\exp\left[-\beta V(r)\right]\right].
\end{aligned}$$

The first term in (4.1.18) is then equal to

$$\begin{aligned}
\frac{1}{k_B T^2} \quad & \left[\int dq_1 \ldots \int dq_{N-1} \exp\left[-\beta {\sum_{ij}}' V_{ij}\right] \int_{r_0}^{r} dr \left(\frac{dV}{dr}\right)\exp\left[-\beta V(r)\right]\right. \\
& \left. + \int dq_1 \ldots \int dq_{N-1} \exp\left[-\beta {\sum_{ij}}' V_{ij}\right] \int_{r_0}^{r} dr \sum_{ij} V_{ij}\left(-\beta\frac{dV}{dr}\right)\exp\left[-\beta V(r)\right]\right].
\end{aligned}$$

We substitute these expressions of the partials of $Q(r,T)$ into equation (4.1.12) to get the first term in (4.1.12) as

$$\begin{aligned}
\frac{\partial Q}{\partial r}\frac{1}{Q^2}\frac{\partial Q}{\partial T} &= \frac{\partial Q/\partial r}{Q}\frac{\partial Q/\partial T}{Q} \\
&= \frac{\int dq_1 \ldots \int dq_N \left(-\beta\frac{dV}{dr}\right)\exp\left[-\beta\sum_{ij} V_{ij}\right]}{Q} \\
&\quad \times \frac{\int dq_1 \ldots \int dq_N \left(\frac{\sum_{ij} V_{ij}}{k_B T^2}\right)\exp\left[-\beta\sum_{ij} V_{ij}\right]}{Q} \\
&\quad + \mathbb{I}\frac{\partial Q/\partial T}{Q^2}
\end{aligned} \tag{4.1.21}$$

or

$$\frac{\partial Q}{\partial r}\frac{1}{Q^2}\frac{\partial Q}{\partial T} = \left\langle -\beta\frac{dV}{dr}\right\rangle \left\langle \frac{1}{k_B T^2}\sum_{ij} V_{ij}\right\rangle + \frac{\mathbb{I}}{Q}\frac{\partial Q/\partial T}{Q}. \tag{4.1.22}$$

Using equations (4.1.19) and (4.1.20), the second term in (4.1.12) is then

$$\begin{aligned}
\frac{1}{Q}\frac{\partial^2 Q}{\partial r \partial T} &= \frac{1}{k_B T^2}\left[\frac{\int dq_1 \ldots \int dq_N \frac{dV(r)}{dr} \exp\left[-\beta \sum_{ij} V_{ij}\right]}{Q} + \left\langle \sum_{ij} V_{ij} \right\rangle^{\star} \frac{\mathbb{I}}{Q}\right. \\
&\left. + \frac{\int dq_1 \ldots \int dq_N \left[\sum_{ij} V_{ij}\left(-\beta \frac{dV}{dr}\right)\right] \exp\left[-\beta \sum_{ij} V_{ij}\right]}{Q}\right] \\
&= \frac{1}{k_B T^2}\left[\left\langle \frac{dV(r)}{dr} \right\rangle - \left\langle \sum_{ij} V_{ij} \beta \frac{dV}{dr} \right\rangle + \sum_{ij} \langle V_{ij} \rangle^{\star} \frac{\mathbb{I}}{Q}\right]. \qquad (4.1.23)
\end{aligned}$$

Substituting (4.1.22) and (4.1.23) into (4.1.12) we get

$$\begin{aligned}
\frac{\partial^2 \ln Q(r,T)}{\partial r \partial T} &= \beta \left\langle \frac{dV}{dr} \right\rangle \frac{1}{k_B T^2} \sum_{ij} \langle V_{ij} \rangle - \frac{\mathbb{I}}{Q} \frac{\partial Q / \partial T}{Q} \\
&+ \frac{1}{k_B T^2}\left[\left\langle \frac{dV}{dr} \right\rangle - \beta \sum_{ij} \left\langle V_{ij} \frac{dV}{dr} \right\rangle + \sum_{ij} \langle V_{ij} \rangle^{\star} \frac{\mathbb{I}}{Q}\right]
\end{aligned}$$

or

$$\begin{aligned}
\frac{\partial^2 \ln Q(r,T)}{\partial r \partial T} &= \frac{1}{k_B T^2}\left[\beta \left\langle \frac{dV}{dr} \right\rangle \sum_{ij} \langle V_{ij} \rangle - \left(k_B T^2 \frac{\partial Q/\partial T}{Q} - \sum_{ij} \langle V_{ij} \rangle^{\star}\right) \frac{\mathbb{I}}{Q}\right. \\
&\left. + \left\langle \frac{dV}{dr} \right\rangle - \beta \sum_{ij} \left\langle V_{ij} \frac{dV}{dr} \right\rangle \right] \\
&= \frac{1}{k_B T^2}\left\{\left\langle \frac{dV}{dr} \right\rangle - \beta \sum_{ij} \left[\left\langle V_{ij} \frac{dV}{dr} \right\rangle - \langle V_{ij} \rangle \left\langle \frac{dV}{dr} \right\rangle\right]\right. \\
&\left. - \left(k_B T^2 \frac{\partial Q/\partial T}{Q} - \sum_{ij} \langle V_{ij} \rangle^{\star}\right) \frac{\mathbb{I}}{Q}\right\}.
\end{aligned}$$

Which can be written as

$$\frac{\partial^2 \ln Q(r,T)}{\partial r \partial T} = \frac{1}{k_B T^2}\left\{\left\langle \frac{dV(r)}{dr} \right\rangle - \frac{d}{dr}\left(\beta \sum_{ij} \Gamma_{ij}\right) - \left[\sum_{ij} \left(\langle V_{ij} \rangle - \langle V_{ij} \rangle^{\star}\right)\right] \frac{\mathbb{I}}{Q}\right\}, \qquad (4.1.24)$$

where

$$k_B T^2 \frac{\partial Q/\partial T}{Q} = \sum_{ij} \langle V_{ij} \rangle ,$$

and

$$\Gamma_{ij} = \langle V(r_{ij}) V(r) \rangle - \langle V(r_{ij}) \rangle \langle V(r) \rangle . \tag{4.1.25}$$

From relation (4.1.20) the last term in (4.1.24) is

$$\sum_{ij} \left(\langle V_{ij} \rangle - \langle V_{ij} \rangle^{\star} \right) \frac{\exp\left[-\beta V(r)\right]}{\int_{r_0}^{r} dr \exp\left[-\beta V(r)\right]} .$$

In our model $V(r) = -\text{constant}/r^6$. Then the integral in the denominator becomes $\int_{r_0}^{r} dr \exp[\text{constant}/r^6]$ which diverges for $|r_0| << 1$. Since $\exp\left[-\beta V(r)\right] \rightarrow 1$ as $r \rightarrow \infty$ we have the result that $\mathbb{I}/Q \rightarrow 0$ as $r \rightarrow \infty$ for our choice of potential and parallel arrangement of dipoles. For a coulombic potential we let $N >> 1$ which will lead to the result that

$$\langle V_{ij} \rangle = \langle V_{ij} \rangle^{\star}$$

that is

$$\lim_{N \rightarrow \infty} \langle V_{ij} \rangle^{\star} = \langle V_{ij} \rangle .$$

This would cause the last term in (4.1.24) to vanish. Therefore, in the limit as N tends to infinity, the last term in (4.1.24) will tend to zero. This makes the application of (4.1.24) to solid materials possible. But since the force between atoms in solids is coulombic, equation (4.1.13) cannot be applied.

Equation (4.1.24) is then

$$\frac{\partial^2 \ln Q(r,T)}{\partial r \partial T} = \frac{1}{k_B T^2} \left[\left\langle \frac{dV}{dr} \right\rangle - \frac{d}{dr} \left(\beta \sum_{ij} \Gamma_{ij} \right) \right] . \tag{4.1.26}$$

From equations (4.1.9), (4.1.10), and (4.1.11) we get

$$\tau = \frac{1}{\cos\theta}\left[\left\langle\frac{dV}{dr}\right\rangle - \frac{d}{dr}\left(\beta\sum_{ij}\Gamma_{ij}\right)\right] \tag{4.1.27}$$

From Plischke and Bergersen [22] the rigorous expression for the magnetic susceptibility is

$$\chi_m = \beta\sum_j\Big(\langle\sigma_j\sigma_0\rangle - \langle\sigma_j\rangle\langle\sigma_0\rangle\Big), \tag{4.1.28}$$

where

$$\begin{aligned} \sigma_j &= \text{the quantum spin of the } j^{\text{th}} \text{ site on a lattice} \\ \sigma_0 &= \text{the spin site for which we calculate the susceptibility.} \end{aligned}$$

If in place of σ_0 and σ_j in (4.1.28) we have $V(r)$ and $V(r_{ij})$, respectively, we recover equation (4.1.25). Because of this analogy in form, we interpret the quantity $\beta\sum_{ij}\Gamma_{ij}$ as the bond susceptibility or "bond strength", that is, how well the dark (k^{th}) molecule is "glued to" the entire chain of molecules. Hence, we introduce the expression $\chi_b \equiv \beta\sum_{ij}\Gamma_{ij}$ and call it the bond strength. The derivative of χ_b which appears in (4.1.27) is then the bond's response to the applied force $\left\langle\frac{dV}{dr}\right\rangle$. Therefore, we rewrite the equivalent surface tension (4.1.27) as

$$\tau = \frac{1}{\cos\theta}\left[\left\langle\frac{dV}{dr}\right\rangle - \frac{d}{dr}\chi_b\right] \tag{4.1.29}$$

with

$$\chi_b = \beta\sum_{ij}\Big(\langle V(r_{ij})V(r)\rangle - \langle V(r_{ij})\rangle\langle V(r)\rangle\Big)$$

as the bond strength term. Equation (4.1.29) was derived from (4.1.9) using relation (4.1.10). We want to point out some of the shortcomings of (4.1.29). From relation (4.1.10) we have

$$Q(r,T) = \int dq_1 \ldots \int dq_{N-1} \exp\left[-\beta \sum_{ij}{}' V_{ij}\right] \int_{r_0}^{r} dr \exp\left[-\beta V(r)\right].$$

While the integral $\int_{r_0}^{r} dr \exp\left[-\beta V(r)\right]$ serves us in getting the extra term in (4.1.24) to converge to zero, it is simultaneously giving us a mathematical problem in the sense that the lower limit is making the integrand diverge. The way to remedy that is to let the lower limit of the integral be small enough to make the extra term in (4.1.24) vanish yet large enough to make the configuration function $Q(r,T)$ converge. We choose this lower limit to be the equilibrium intermolecular distance at the minimum potential energy, that is, r_0 in Figure (16) shown below:

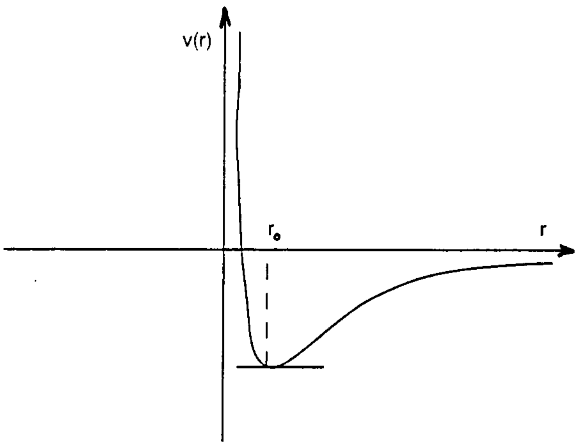

Figure 16: The potential energy curve between two molecules.

Therefore,

$$Q(r,T) = \int dq_1 \ldots \int dq_{N-1} \exp\left[-\beta \sum_{ij}{}' V_{ij}\right] \int_{r_0}^{r} dr \exp\left[-\beta V(r)\right].$$

With this form of Q the expressions in (4.1.29) are all convergent. Equation (4.1.29) can be expressed in words by

$$\begin{aligned}\text{Surface tension} &= (\text{attenuation factor}) \\ &\qquad \times(\,-\,\text{applied force} + \text{bond's strength response}).\end{aligned}$$

To interpret the expressions on the right hand side of (4.1.29), it is crucial that the signs of the two terms are opposite. The first term is the Newtonian pulling or pushing force. The two terms are then due to the applied force and the chemical bonds response to it in the sense of bond "stretch" or "compression". A very important observation here is the consequence of choosing the specific arrangement of dipoles. The parallel arrangement we chose gave us a potential in the form $V(r) = -4\alpha\mu\mu/r^6$, hence $\langle dV/dr\rangle$ is a negative force. $\beta\sum_{ij}\Gamma_{ij}$ is positive for all arrangements parallel or anti-parallel, which means that at higher temperatures the bond strength contribution to the magnitude of τ is lessened, a result we expect to be true since higher temperatures are a reflection of higher kinetic energy values, which will cause the molecules to have more difficulty bonding with each other.

Another observation here is that equation (4.1.9) is a basic physics equation as is equation (4.1.29). That is, basic physics lead to basic physics. This encourages us to think of (4.1.9) as the right approach to the problem, because the statement expressed in (4.1.29) is a natural consequence of the situation encountered in Figure (15).

4.2 Applications

Our expression for the surface tension includes many special cases. One situation where we encounter such an expression is the differential equation of surface waves. We would like to consider this differential equation without assuming small amplitudes, Since we can consider the water surface to be a sheet or a membrane, we can define a mass per unit area. Accordingly we let

$$\begin{aligned} \sigma &= \text{Mass per unit area} \\ dA &= \text{Equilibrium area} \\ dS &= \text{Actual stretched area} \end{aligned}$$

If we define the potential energy as the work done in stretching the membrane we have from Fetter & Walechka [11]

$$\mathcal{V}dA = \tau(dS - dA) \tag{4.2.30}$$

where

$$\begin{aligned} \tau &= \text{Surface tension or surface energy density} \\ \mathcal{V} &= \text{potential energy density} \end{aligned}$$

If we let $\hat{z}$ be the unit vector in the direction of the positive z coordinate, $\hat{n}$ the unit normal to the surface, and $u(x, y)$ be the vertical displacement of the membrane from its equilibrium configuration, then we have (see [11])

$$\hat{n} = \frac{\hat{z} - \vec{\nabla}u}{\left[1 + (\vec{\nabla}u)^2\right]^{1/2}} \tag{4.2.31}$$

which leads to

$$\hat{z} \cdot \hat{n} = \frac{1}{\left[1 + (\vec{\nabla}u)^2\right]^{1/2}}. \tag{4.2.32}$$

Let $d\vec{s} = \hat{n}dS$. Since the equilibrium area dA is the projection of this vector in the z direction, we have $dA = \hat{z} \cdot (\hat{n}dS)$. Equation (4.2.30) then becomes

$$\mathcal{V}dA = \tau dA\left\{[1 + (\vec{\nabla}u)^2]^{1/2} - 1\right\}. \tag{4.2.33}$$

The kinetic energy is

$$\mathcal{T}dA = \frac{1}{2}\sigma dA\left(\frac{\partial u}{\partial T}\cdot\right)^2 \tag{4.2.34}$$

Combining equations (4.2.33) and (4.2.34) we find

$$\mathcal{L} = \frac{1}{2}\sigma(x)\left(\frac{\partial u}{\partial t}\right)^2 - \tau\left\{[1 + (\vec{\nabla}u)^2]^{1/2} - 1\right\}, \tag{4.2.35}$$

which is equivalent to the Lagrangian density for a vibrating membrane. The Euler-Lagrange equation is

$$\frac{\partial}{\partial t}\frac{\partial\mathcal{L}}{\partial(\partial u/\partial t)} + \frac{\partial}{\partial x}\frac{\partial\mathcal{L}}{\partial(\partial u/\partial x)} + \frac{\partial}{\partial y}\frac{\partial\mathcal{L}}{\partial(\partial u/\partial y)} - \frac{\partial\mathcal{L}}{\partial u} = 0. \tag{4.2.36}$$

If we calculate the partials in the Euler-Lagrange equation separately, we obtain

$$\begin{aligned}
\frac{\partial\mathcal{L}}{\partial(\partial u/\partial t)} &= \sigma\left(\frac{\partial u}{\partial t}\right) \\
\frac{\partial}{\partial t}\frac{\partial\mathcal{L}}{\partial(\partial u/\partial t)} &= \frac{\partial}{\partial t}\sigma\left(\frac{\partial u}{\partial t}\right) = \sigma\frac{\partial^2 u}{\partial t^2} \qquad\qquad (\sigma = \sigma(x)) \\
\frac{\partial\mathcal{L}}{\partial u_x} &= -\tau\frac{\partial u/\partial x}{[1 + (\vec{\nabla}u)^2]^{1/2}} \\
\frac{\partial\mathcal{L}}{\partial u_y} &= -\tau\frac{\partial u/\partial y}{[1 + (\vec{\nabla}u)^2]^{1/2}} \\
\frac{\partial\mathcal{L}}{\partial u} &= 0.
\end{aligned}$$

Substituting these expressions in equation (4.2.36) we get

$$\sigma\left(\frac{\partial^2 u}{\partial t^2}\right) = \vec{\nabla}\cdot\left\{\frac{\tau(\vec{x})}{[1+(\vec{\nabla}u)^2]^{1/2}}\vec{\nabla}u\right\}. \tag{4.2.37}$$

In the case of pure water, σ can be taken to be constant because this implies molecular replacements from the volume region as the surface is stretched. This was demonstrated in Figures 1 and 2.

Equation (4.2.37) is the unapproximated wave equation of surface waves and we note the significance of the surface tension equation. If the surface tension is constant, equation (4.2.37) reduces to

$$\frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} = \vec{\nabla}\cdot\left\{\frac{\vec{\nabla}u}{[1+(\vec{\nabla}u)^2]^{1/2}}\right\}, \tag{4.2.38}$$

where $c^2 \equiv \tau/\sigma$ is the velocity of propagation. However, if the surface tension is not constant then that will add to the complexity in the form of equation (4.2.37).

To put equation (4.2.37) into a form appropriate for describing surface water waves, we let $u = \xi$ be the displacement from the equilibrium surface. Then equation (4.2.37) would be

$$\sigma\left(\frac{\partial^2 \xi}{\partial t^2}\right) = \vec{\nabla}\cdot\left\{\frac{\tau}{[1+(\vec{\nabla}\xi)^2]^{1/2}}\vec{\nabla}\xi\right\}. \tag{4.2.39}$$

From the inviscid flow equation we have

$$\frac{d\vec{v}}{dt} = \frac{\partial\vec{v}}{\partial t} + (\vec{v}\cdot\vec{\nabla})\vec{v} = -\frac{\vec{\nabla}P}{\rho} + \vec{f},$$

where $\vec{f} = -\vec{\nabla}\Phi(\vec{x})$ and $\Phi(\vec{x}) = \phi z$.

Furthermore, we assume the fluid to be irrotational so that $\vec{\nabla}\times\vec{v} = 0$, which will lead to the equilibrium expression $\vec{v} = -\vec{\nabla}\phi$. Using the identity $(\vec{v}\cdot\vec{\nabla})\vec{v} = \vec{\nabla}(\frac{1}{2}v^2) - \vec{v}\times(\vec{\nabla}\times\vec{v})$, the flow equations become

$$-\frac{\partial\phi}{\partial t} + \frac{1}{2}v^2 + \frac{P}{\rho} + \phi z = 0.$$

Let P_0 denote the atmospheric pressure and $\bar{\phi} = \phi - P_0 t/\rho$. From Fetter & Walecka [11], equation (4.2.34) gives

$$\rho\phi\xi - \vec{\nabla}\cdot\left\{\frac{\tau}{[1+(\vec{\nabla}\xi)^2]^{1/2}}\vec{\nabla}\xi\right\} = \rho\frac{\partial\bar{\phi}}{\partial t} - \frac{1}{2}\rho v^2 \tag{4.2.40}$$

as the unapproximated boundary condition imposed by surface waves.

As the above equations demonstrate the application of our mathematical models of the water surface tension equation, we would like now to demonstrate the application of our physical model of water surface tension to the phenomenon of the spherical shape of water droplets.

We know that the shape of the water liquid drop depends on the surface on which it resides. Since we derived the surface tension from electromagnetic forces, we can explain the fact that water drops are more spherical on a non-polar surface. An example of a non-polar surface is Teflon [1] which is the trade name for polytetrafluoroethylene (TFE). Teflon is a highly ordered polymer made of chains of CF_2 with strong C-F bond. It is a stiff, rigid insulator with low coefficient of friction. Since it is an insulator, it is electrically inert. Our model explains the spherical shape water droplets take on a teflon surface. In the case of a polar surface the electromagnetic interaction between the water droplet and the polar surface flattens the water droplet.

5 Critical Velocity of Interactions

In this chapter we combine the results of chapters 3 and 4 to come up with an interesting equation. We described the motion of scattered particles by water beams in chapter 3, but we did not give a theoretically based, physical analysis concerning the actual cause of the scattering of the residing particles by the streamlines. Besides the transfer of momentum that we mentioned in chapter 3, there are other factors that contribute to the process of scattering. This argument is supported by experiment, where residing particles are not always scattered by water beams. It turns out that there are certain critical values at which the actual scattering occurs. In chapter 3 we described the scattering in terms of a force delivered by the streamlines at the instant of scattering, and we expect one of these critical values to be a dynamical variable on which the force transferring the momentum depends. From (3.1.16) this dynamical variable is either the pressure or the velocity of the scatterer, here the water beam. From the perspective of the residing particles, it is the chemical bonding between the particles that contributes to the determination of the critical value of that dynamical variable. Since we are talking about "mud", which is a mixture of dust particles with water, the chemical forces between the water molecules also play an important role in determining the critical value. It is here that the results of chapter 4 are applied. The forces between the water molecules in the muddy bottom are similar in form to the forces causing water surface tension. Let us now start the analysis starting with Figure (1) in chapter 3.

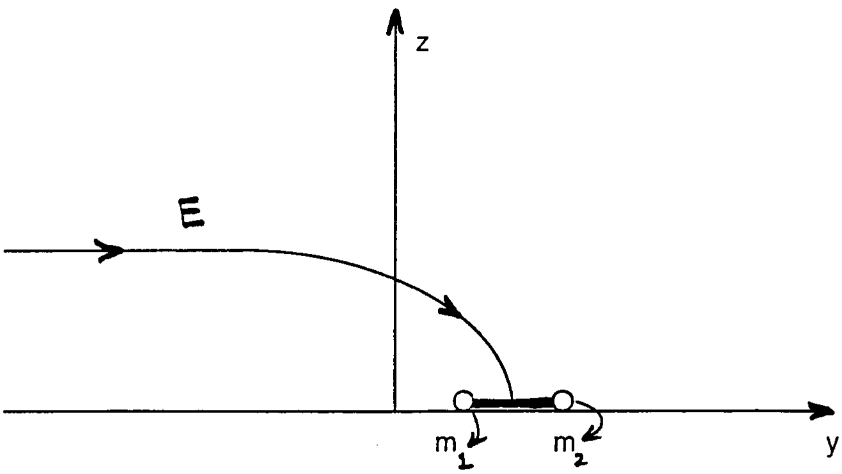

Figure 17: Showing a streamline with energy E' interacting with residing particles.

Let E' and E be the energy of the bond between the masses m_1 and m_2 shown in the figure and the energy of the streamline scattering with the bond, respectively. By definition of the critical energy, E', if $E < E'$, then the bond holding m_1 and m_2 together will not break. However, for $E \geq E'$, the bond will break and the two particles will fly apart. We deduce the nature of this bond from the results of chapter 4. Now let ε be the internal energy of a streamline per unit mass and $\mathscr{H}$ be the enthalpy per unit mass of streamline. Then from Landau and Lifshitz [19], we have

$$E = [\rho\vec{v}(\frac{1}{2}v^2 + \mathscr{H})] \cdot \hat{n} \tag{5.0.1}$$

where adiabatic flow is assumed and where $\hat{n} = -\hat{z}$ at the instant of collision. The quantity $\mathscr{H}$ in (5.0.1) is the enthalpy (heat function per unit mass) while E is the energy flux density vector component along $\hat{n}$ of the scattering streamline.

If we define ε to be the internal energy per unit mass and write the enthalpy as

$\mathscr{H} = \varepsilon + P/\rho$ with P as the pressure and ρ as the mass density,then we can write

$$\rho\vec{v}(\frac{1}{2}v^2 + \mathscr{H}) = \rho\vec{v}(\frac{1}{2}v^2 + \varepsilon) + P\vec{v}. \tag{5.0.2}$$

Also from Landau and Lifshitz [19] we have the equality

$$-\frac{\partial}{\partial t}\int_V(\frac{1}{2}\rho v^2 + \rho\varepsilon)d^3x = \oint_S \rho\vec{v}(\frac{1}{2}v^2 + \mathscr{H}) \cdot d\vec{a}. \tag{5.0.3}$$

According to (5.0.3), E is the rate of the streamline's energy dissipation through the surface $\hat{n}da$. From equation (5.0.2) we can write

$$\begin{aligned} E &= [\rho\vec{v}(\frac{1}{2}v^2 + \varepsilon) + P\vec{v}] \cdot \hat{n} \\ &= \rho v_n(\frac{1}{2}v^2 + \varepsilon) + Pv_n \\ &= \rho v_n[\frac{1}{2}(v_n^2 + v_\perp^2) + \varepsilon] + Pv_n \\ &= \frac{1}{2}\rho v_n^3 + \frac{1}{2}\rho v_n v_\perp^2 + \rho v_n\varepsilon + Pv_n \\ &= \frac{1}{2}\rho v_n^3 + (\frac{1}{2}\rho v_\perp^2 + \rho\varepsilon + P)v_n \end{aligned} \tag{5.0.4}$$

where $\vec{v} \cdot \hat{n} = v_n$ is the only non zero component at the instant of the collision or the energy transfer. Accordingly, we set $v_\perp = 0$ and (5.0.4) becomes

$$E = \frac{1}{2}\rho v_n^3 + (\rho\varepsilon + P)v_n. \tag{5.0.5}$$

Therefore, E is a cubic equation in v_n and (5.0.5) is our mathematical expression for the energy of the scattering streamline. The next step is to find an expression for E', the energy of the chemical bond between the particles. Figure (18) below demonstrates for us the chemical bonding between the particles at a microscopic level.

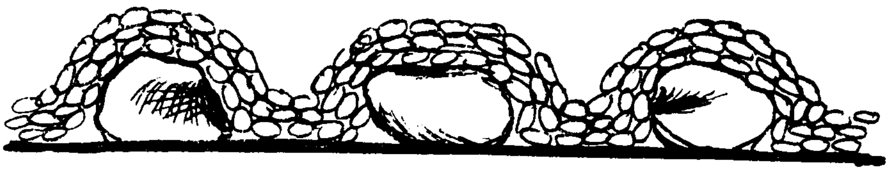

Figure 18: Showing dust particles bonded by chain of water molecules.

In Figure (18) we approximate a water molecule to be the one given in Figure (15) of chapter 4. We assume no interactions between the particles (non cohesive sediments) and no interactions between the particles and the molecules. Consequently a single dust particle is held down by a layer of water molecules with permanent dipoles. Recalling the distinct arrangements we showed in chapter 4 following Figure (15), it becomes clear that the arrangement shown in Figure (18) is the one that generates water surface tension. From equation (4.1.1), the potential energy between two dipoles in electrostatic units is

$$U(r_{12}) = -\frac{2\mu\mu'}{r_{12}^3} \tag{5.0.6}$$

where $\mu' = 2\alpha\mu/r^3$. Equation (5.0.6) becomes

$$U(r_{ij}) = -\frac{4\alpha\mu^2}{r_{ij}^6}, \tag{5.0.7}$$

where α is the polarizability we introduced in chapter 4. Now consider a single layer of molecules in Figure (18). From Figure (18) and equation (5.0.7) the energy of the kth layer is

$$E'_k = -\sum_{\substack{i,j \\ i \neq j}} \frac{4\alpha\mu^2}{r_{ij}^6}, \quad k = 1, \ldots, N, \tag{5.0.8}$$

so the total energy of N layers is then

$$E' = \sum_{m=1}^{N} E'_m = NE'_k, \tag{5.0.9}$$

where for simplicity we assumed no interaction between the layers. N is chosen such that $0 < E' < \infty$. By the equations (5.0.5), (5.0.8), and (5.0.9), the condition $E \geq E'$ gives the inequality

$$\frac{1}{2}\rho v_n^3 + (\rho\varepsilon + P)v_n \geq -N \sum_{i,j} \frac{4\alpha\mu^2}{r_{ij}^6}, \tag{5.0.10}$$

which leads to $inf\{\frac{1}{2}\rho v_n^3 + (\rho\varepsilon + P)v_n\} = -N\sum_{i,j}\left(4\alpha\mu^2/r_{ij}^6\right)$, where the infimum is taken over the v_n.

The critical value where the infimum is assumed is denoted by v_{nc}. With the equality we let $v_n \equiv v_{nc}$ and define it to be the critical velocity at which the scattering take place. Equation (5.0.10) then becomes

$$\frac{1}{2}\rho v_{nc}^3 + (\rho\varepsilon + P)v_{nc} = -N\sum_{\substack{i,j \\ i\neq j}} \frac{4\alpha\mu^2}{r_{ij}^6}$$

or

$$v_{nc}^3 + \frac{2(P+\varepsilon\rho)}{\rho}v_{nc} = -\frac{8N}{\rho}\sum_{\substack{i,j \\ i\neq j}} \frac{\alpha\mu^2}{r_{ij}^6}. \tag{5.0.11}$$

Note the direct proportionality in (5.0.11) between v_{nc} and N. There is also an inverse proportionality between v_{nc} and T. This is because the polarizability α can be written, from J. C. Slater [26], as

$$\alpha = \alpha_0 + \frac{\mu^2}{3k_B T} \tag{5.0.12}$$

where α_0 is the polarization due to electronic fluctuation.

Equation (5.0.11) defines v_{nc} and gives us a direct way of calculating v_{nc} which is what we sought in this chapter. The proportionalities between the variables of (5.0.11) makes it even more interesting. Taking (5.0.11) to a continuous limit we write

$$v_{nc}^3 + \frac{2(P+\varepsilon\rho)}{\rho}v_{nc} = -\frac{8N}{\rho}\int_0^L dx \int_0^L dy \frac{\alpha\mu^2}{[r(x,y)]^6} \tag{5.0.13}$$

with $|L| \gg 1$.

If we are dealing with cohesive sediments then the energy due to coulombic force between the sediments should be added to the right hand side of (5.0.13).

6 Discussion

The equations we presented in chapters 3, 4, and 5 were very general and complex. In particular,in developing the equations for the hydrodynamical, continuous mass transport, physical situation considered forced the equations to become complex. Our intention, however, was to give hydrodynamical equations of motion and hydrodynamical forces that had never been considered before and hence give an initial step toward a better theoretical approach to the problem of dust particles in hydrodynamical flows. To describe rotational flows effectively, we wrote the fluid velocity as

$$\vec{v}(\vec{x},t) = -\vec{\nabla}\phi(\vec{x},t) + \vec{\nabla} \times \vec{A}(\vec{x},t). \tag{6.0.1}$$

The curl of the vector potential $\vec{A}$ would then be responsible for the nonzero vorticity. From the Navier-Stokes equations, we found that $\vec{A}$ satisfied

$$\rho\frac{\partial^2 \vec{A}}{\partial t^2} - \eta\nabla^2\vec{A} = C_0\frac{\partial\left(\rho\vec{\mathscr{F}}_\perp\right)}{\partial t}. \tag{6.0.2}$$

where ρ is the fluid density, η is the drag force coefficient, $\vec{\mathscr{F}}$ is a vector quantity with velocity sqaured dimensions, and C_0 is a constant of proportionality. By equating the right hand side of(6.0.2) to zero, we were able to get a homogeneous wave eqaution for $\vec{A}$. Consequently, $\vec{A}$ was an oscillatory function, which in turns gave us an oscillatory viscous drag force explaining the phenomenon of sand ripples in deserts and on ocean beaches. This, as we mentioned in the introduction, is the first concerete and successful explanation given in explaining the phenomenon of ripples. Another new and successful explanation of a physical phenomenon we achieved by defining $\vec{A}$ was the eqaution

$$\frac{\partial h}{\partial t} = \frac{-(2\pi)^2}{\lambda_y} A_0 e^{i(\vec{k}\cdot\vec{x}-\omega t)}. \tag{6.0.3}$$

Where h is the fluid height at a solid boundary, λ_y is the wavelength of the incoming water, and A_0 is the amplitude of $\vec{A}$. Equation (6.0.3) describes the motion of a fluid surface at a solid boundary. The inverse proportionality between λ_y and h and the oscillatory behavior expressed in the exponential are consistent with what is observed physically. This equation can be solved easily for h and values for λ_y and A_0 can be substituted to get a numerical value for h.

The generality of the right hand side of equation (6.0.2) allows us to get expressions for drag forces and hence give an initial step in a theoretical description of forces involved during the turbulent motion of a fluid. One such expression we obtained was the equation

$$\frac{\partial\left(\rho\vec{\mathscr{F}}_{\perp}\right)}{\partial t} = \vec{\nabla}(\eta v^2). \tag{6.0.4}$$

All these consequences of introducing the quantity $\vec{A}$ supports the fact that turbulent flow in continuous media is due to the presence of the second term in equation (6.0.1). This is a new step and, we believe, a better approach to the understanding of turbulent phenomena. We do not have to deal with nonlinear dynamics and the equations are very concrete in their description of continuous matter flows. Other studies about turbulence and rotational flow must include $\vec{A}$ and it's role in causing a turbulent flow.

An additional advantage of our force equation due to fluid motion is that it is expressed in terms of a Lagrangian density which we know to be not unique. Our way of describing continuous mass motion in fluids was to introduce equations for discrete masses disturbed by discrete streamlines then take the discrete equations to a continuous limit. This gives a guideline and a clear procedure for other researchers on how to obtain equations for continua. To include the many interactions between the mass particles, we applied the propagator theory to the continuous form of our equations. This step was one of the most successful steps we achieved. A very interesting calculation for future work is to calculate a two particle interaction propagator and substitute it into the continuous equations.

The first equation we introduced for surface tension involved the van der Waals potential. The second equation was expressed in terms of a general potential. We understand that our model was one dimensional , however, for future search a two dimensional model can be developed using our results as a guide for finding significant quantities, specifically, the bond strength term.

The equation we introduced in chapter 5 is a cubic and a simple equation. It defines a critical streamline velocity at which an interaction between the fluid in motion and the residing particles can occur. We realize that this is the first attempt to define a physical quantity that the interaction between the fluid and the residing particles depend on. Since experiments of this kind are not too costly, values of critical velocity obtained by experiments can be compared with our theoretical value. A bonding between the particles that considers all the attraction forces can be studied in future works in an attempt to obtain a theoretical critical velocity value agreeing with the experimental one.

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